KU LEUVEN

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Nonlinear Control Systems

A "State-Dependent (Differential) Riccati Equation" Approach

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Li-Gang (Charles) Lin

Dissertation presented in partial fulfillment of the requirements for the double degrees of Doctor in Engineering from KU Leuven and NCTU

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Preface

The road to truth never ends, but lined the entire way with surprises, excitements, and lots of fun.

The research of Nonlinear Control Systems fascinates me a lot, and I feel very grateful that I could pursue my interest till this stage. Since childhood, I like mathematics and physics very much, but know nothing about control until college. After a first glance during a course in the Linear Control Systems, I was addicted to it (I don't smoke, but I suppose it is a good way to express such a feeling). Through all the days surrounded by "control", now I am finishing my joint/double PhD program with an emphasis on the state-dependent (differential) Riccati equation scheme (SDRE/SDDRE) for the nonlinear control systems.

Regarding the joint/dual/double PhD program, such program has been quite popular worldwide for years, and some other universities (like Harvard, Yale, MIT, UIUC, Duke, NUS, TU Delft, and Peking University) also appreciate its importance and thus include it as well. Through the training of such program, the student could easily interact with scholars worldwide, learn from various traditions/experiences, and obtain a global/international view from several universities, which are also beneficial for the university in the aspect of diversity, and international visibility. Gratefully, I really appreciate this opportunity, and thank you KU Leuven and National Chiao Tung University (NCTU).

It is a blessing for me to achieve this stage, and I may not be able to express my gratitude and acknowledgment sufficiently. Foremost of all, I would like to thank my family, i.e. my father, my mother, and my little brother, for their unlimited/uncountable support and care. They are always there, on my side, to listen to me and help me deal with difficulties. Gratefully, I will try my best to repay everything to you, my family.

All the way till now, there are so many persons I would like to express my gratitude. For the concern of pages, I am not willing but have to restrict to the stage of the joint/double PhD program. At first, I was trained in NCTU, and under the supervision by Prof. Yew-Wen Liang. He taught me a lot about control, with various mathematical preliminaries. It is worth mentioning that he introduced the SDRE scheme to me, which became the main topic of my thesis. Besides the professional area, He also taught me about how to behave, helped me tackle troubles, and gave me enough freedom to learn more. Gratefully, I with all the best to you. Moreover, I would like to thank all other professors that I took courses from, which are really important and indispensable. Gratefully, thank you Prof. S.-M. Chang, Prof. M.-C. Li, Prof. D.-C. Liaw, Prof. S.-K. Lin, Prof. Y.-P. Chen, Prof. T.-S. Sang, Prof. C.-C. Teng, and Prof. P.-Y. Wu (alphabetically in the family name).

Next, I was trained in KU Leuven. With great thanks to my supervisor Prof. dr. ir. Joos Vandewalle, I really learn a lot in the Nonlinear System Analysis. He led me to think about problems in different aspects, and helped me dig out any possible potential of the SDRE/SDDRE scheme, from both the theoretical and application viewpoints. Through all the excellent interactions, I really appreciate you time and efforts. Gratefully, I wish all the best to you, and enjoy your retirement in the best of health. Het ga je goed! Still, I would like to thank all professors and researchers during any cooperation/interaction, your ideas and opinions stimulate the progress of the SDRE/SDDRE scheme. Gratefully, thank you Prof. M. Diehl, Prof. S. Gros, Prof. E. J. M. Reyes, Prof. C. R. V. Seisdedos, Prof. L .V. Seisdedos, Dr. J. Stoev, Prof. J. Swevers, Prof. H. Van Brussel and Dr. S. Vandenplas. Furthermore, I had a great time with my colleagues, and really enjoyed all the company. Gratefully, thank you R. Castro, Dr. H.-M. Chao, Dr. Y. Fan-Chiang, Dr. Y.-L. Feng, G. Horn, Dr. X.-L. Huang, J. Gillis, Dr. A. Kozma, Dr. S. V. Kungurtsev, Dr. R. Langone, R. Ribas Manero, A. Mohammadi, Dr. R. Quirynen, X. Wang, Dr. L. Weng, Dr. L. Zhang, Dr. X.-R. Zhang, and Dr. X.-Z. Zheng (alphabetically in the family name).

Last but not least, many thanks to all the members in my examination committee, the chair Prof. L. Froyen, the supervisors Prof. Y.-W. Liang and Prof. J. Vandewalle, and the assessors Prof. Y.-P. Chen, Prof. C.-S. Hsieh, Prof. W. Michiels, Prof. J. Suykens, and Prof. J. Van Impe (alphabetically in the family name). Your valuable comments are of great importance and definitely constructive. If without your efforts, I could not complete the thesis. Gratefully, thanks very much and I wish you all the best.

With all due respects, if I carelessly overlook to list your name above, please accept my sincere apologies, and your kind help is definitely beneficial and essential. Gratefully, thank you from the bottom of my heart.

Leuven, Belgium, 2014
Li-Gang (Charles) Lin

Abstract

In the area of nonlinear control systems, recently the easy-to-implement state-dependent Riccati equation (SDRE) strategy has been shown to be effective by numerous practical applications, possessing collectively many of the capabilities and overcoming many of the difficulties of other nonlinear control methods. Its diverse fields of applications include missiles, aircrafts, satellites, ships, unmanned aerial vehicles (UAV), biomedical systems analysis, industrial electronics, process control, autonomous maneuver of underwater vehicles, and robotics. Due to the great similarity to SDRE, the newly emerged state-dependent differential Riccati equation (SDDRE) approach exhibits great potential from both the analytical and practical viewpoints, and shares most of the benefits of SDRE while differing mainly in the time horizon considered (i.e. finite for SDDRE and infinite for SDRE). However, there is a significant lack of theoretical fundamentals to support all the successful implementations, especially the feasible choice of the possessed design flexibility (namely, the infinitely many factorizations of the state-dependent coefficient matrix) with predictable performance is still under development for both schemes. In this thesis, considering the general finite-order nonlinear time-variant systems, several problems related to the design flexibility are investigated and solved, which appear at the very beginning of the implementation of both schemes. Finally, connections to the literature in various topics of research are established, and the proposed scheme is demonstrated via examples, including real-world applications.

Abbreviations

(in alphabetical order)

AIAA	American Institute of Aeronautics and Astronautics
AIPN	adaptive ideal proportional navigation
ASME	American Society of Mechanical Engineers
AUV	autonomous underwater vehicles
BS	back-stepping
CVD	chemical vapor deposition
DOA	domain of attraction
DOF	degree of freedom
FL	feedback linearization
FTHNOC	finite-time horizon nonlinear optimal control
GAS	globally asymptotically stable
HARV	high alpha research vehicle
HIV	human immunodeficiency virus
HJ	Hamilton-Jacobi
HJB	Hamilton-Jacobi-Bellman
IEEE	Institute of Electrical and Electronics Engineers
IFAC	International Federation of Automatic Control
ISMC	integral-type sliding mode control
ITHNOC	infinite-time horizon nonlinear optimal control
LD	linearly dependent
LHP	left-half plane
LI	linearly independent
LPV	linear parameter varying
LQR	linear quadratic regulator
LTI	linear time-invariant
MICA	missile d'interception et de combat aérien (interception and aerial combat missile)
(N)MPC	(Nonlinear) Model Predictive Control
PBH	Popov-Belevitch-Hautus

PPN	pure proportional navigation
PI	proportional-integral
PMSM	permanent magnet synchronous motor
IPMSM	interior permanent magnet synchronous motor
SAV	space access vehicle
SDDRE	state-dpendent differential Riccati equation
SDRE	state-dpendent Riccati equation
SDC	state-dependent coefficient
UAV	Unmanned aerial vehicles

Nomenclature

(unless otherwise mentioned in this thesis)

$(\cdot)^T$	the transpose of a vector or a matrix
A	$A(\mathbf{x}, t)$
B	$B(\mathbf{x}, t)$, with full column rank
C	$C(\mathbf{x}, t)$, with full row rank
P	$P(\mathbf{x}, t)$
Q	$Q(\mathbf{x}, t)$
R	$R(\mathbf{x}, t)$
$\mathbf{f}$	$\mathbf{f}(\mathbf{x}, t)$
$W^\perp, W_\perp$	Case (i): $W \in \mathbb{R}^{p\times n}$ and $\mathrm{rank}(W) = p < n$ $W^\perp = N(W)$, null space of W, and $W_\perp \in \mathbb{R}^{n\times(n-p)}$ as a selected constant matrix having orthonormal columns and satisfying $WW_\perp = 0$. Clearly, $W^\perp$ is a vector space of dimension $n-p$, and the column vectors of $W_\perp$ form an orthonormal basis of $W^\perp$. Case (ii): $W \in \mathbb{R}^{n\times q}$ and $\mathrm{rank}(W) = q < n$ $W^\perp = \{\mathbf{w}^T \mid \mathbf{w} \in N(W^T)\}$ and $W_\perp \in \mathbb{R}^{(n-q)\times n}$ as a selected constant matrix having orthonormal rows and satisfying $W_\perp W = 0$.
$\mathcal{A}_{\mathbf{xf}}$	$\left\{A_p + K\mathbf{x}_\perp \mid K \in \mathbb{R}^{n\times(n-1)}\right\} \subset \mathbb{R}^{n\times n}$, the sets of A such that $A\mathbf{x} = \mathbf{f}$
$\mathcal{A}^c$	the sets of A such that (A, B) is controllable
$\mathcal{A}^s$	the sets of A such that (A, B) is observable
$\mathcal{A}^o$	the sets of A such that (A, C) is observable
$\mathcal{A}^d$	the sets of A such that (A, C) is detectable
$\mathcal{A}^l$	the sets of A such that (A, C) has no unobservable mode on the LHP and $j\omega$-axis
$\mathcal{A}^i$	the sets of A such that (A, C) has no unobservable mode on the $j\omega$-axis

$\mathcal{A}_{\mathbf{xf}}^{\alpha\beta}$	$\mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^{\alpha} \cap \mathcal{A}^{\beta},\ \alpha = c, s,\ \beta = o, d, l, i$
A_p	$arg \min\limits_{A \in \mathcal{A}_{\mathbf{xf}}} \|\|A\|\|_F = \frac{1}{\|\|\mathbf{x}\|\|^2} \mathbf{f}\mathbf{x}^T$
$\|\| \cdot \|\|$	Euclidean norm
$\|\| \cdot \|\|_F$	Frobenius norm
A_{CL}	$A_{CL}(\mathbf{x}, t)$
$A_{CL}(\mathbf{x}, t)$	$A(\mathbf{x}, t) - B(\mathbf{x}, t)R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t)$, for the SDRE scheme
$\mathbb{R}^{n*}$	$\{\mathbf{x}^T \| \mathbf{x} \in \mathbb{R}^n\}$, the dual space of $\mathbb{R}^n$
$\mathbb{R}^+$	the set of nonnegative real numbers, $\mathbb{R}^+ = [0, \infty) \subset \mathbb{R}$
$\mathbb{R}^-$	the set of negative real numbers, $\mathbb{R}^- = (-\infty, 0) \subset \mathbb{R}$
$L_2(\mathbb{R}^+)$	the Lebesque space, consisting of measurable square-integrable (vector-valued) functions $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^{\mathrm{m}}$, such that $\int_{\mathbb{R}^+} \|\|\mathbf{u}\|\|^2 dt < \infty$
card(A)	the cardinality of a set A

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Chapter 1

Introduction

Around 1960s, the analysis and design for linear systems had been developed to a large extent. However, almost every real and practical system exhibit nonlinear characteristics, and the linear system theory usually applies only within a limited domain around the operation point with respect to the linearized model. Therefore, starting from around 1970s, the control community was paying more and more attention into the design of nonlinear systems, with fruitful contributions such as model predictive control, center manifold theory, singular perturbation analysis, Jacobian linearization, H_∞ control, gain scheduling, feedback linearization (FL), variable structure system, back-stepping (BS), behavioral approach, and dynamic inversion [120, 132, 136, 165, 224, 247]. Although currently no universal formula for all systems, users may choose the one with the most acceptable performance. To apply the above-mentioned control strategies, some restrictions and assumptions are required inherently. For example, BS applies only to the system of the strict-feedback form [132]; while FL requires the system being linearizable through the coordinate transformation and state feedback [224]. Therefore, there often exist some difficulties in applying theoretical results to real applications, not to mention the capabilities of robustness, reliability, and etc.

Concerning the performance of real-world applications, the control community is interested in developing object-oriented, systematic, and easy-to-implement nonlinear control schemes, preferably including the flexibility in tuning between control efforts and steady state error (optimally with respect to some pre-determined criteria). Among these schemes, the state-dependent Riccati equation (SDRE) approach has recently attracted considerable attention and has become a promising and popular synthesis tool over the last decade (e.g.

[40, 51, 54, 77, 153, 178, 255]). Inspired by the SDRE design, the state-dependent differential Riccati equation (SDDRE) approach is progressing promisingly for nonlinear system control (e.g. [105, 111, 113, 194, 203]).

Unlike aiming at infinite-time horizon nonlinear optimal control (ITHNOC) for the SDRE strategy, the SDDRE scheme focuses on the finite-time horizon nonlinear optimal control (FTHNOC). Possessing the additional constraint on final time, the FTHNOC problem seems to be more complicated and challenging than ITHNOC. Associated with FTHNOC (resp., ITHNOC), it is well-known that the Hamilton-Jacobi-Bellman, HJB, (resp., Hamilton-Jacobi, HJ) equation is required to calculate the optimal control law and is in general very difficult to solve [111] (resp., [116]). As a consequence, plenty of methods such as Taylor series based method [237], which inherits the limited domain of convergence of the series, and neural network solution [45], which assumes that trained domain of neural network includes the resulted trajectory, have been developed for the FTHNOC problem. Likewise, there also exist difficulties in solving the ITHNOC problem [54]. Therefore, the goal of the SDDRE (resp., SDRE) control strategy is trying to deal with the FTHNOC (resp., ITHNOC) problem in a more implementable way.

Due to the similarity between SDRE and SDDRE, they both share the following beneficial properties: (1) their intuitive and simple concepts directly adopt the design of the linear quadratic regulator (LQR) at every nonzero state; (2) the designs can directly affect system performance with reasonable outcomes by tuning the state and the control weighting to specify the performance index (e.g., the engineer may modulate the weighting of the system state to speed up the response although at the expense of increased control effort); (3) the schemes possess an extra design degree of freedom (DOF) arising from the non-unique state-dependent coefficient (SDC) matrix representation of the nonlinear drift term, which can be utilized to enhance the controller performance; and (4) the approaches preserve the essential system nonlinearities because they do not truncate any nonlinear terms.

Many practical applications successfully performed by the SDRE strategy have been reported [51], e.g. missile guidance and satellite attitude control, autonomous maneuver of underwater vehicles, systems biology analysis, and robotic manipulation. However, it seems that the achievements in practical applications have outpaced that of theoretical fundamentals, which motivates this thesis to provide more theoretical support. On the other hand, to the authors' knowledge, there are few practical applications using the SDDRE strategy. Due to the great resemblance to the SDRE strategy, the SDDRE scheme is promising and possesses interesting potentials in view of applications.

1.1 Mathematical Preliminaries

This section briefed an overview of the fundamental background materials and mathematical preliminaries for the SDRE scheme, regarding the ITHNOC problem. Due to the similarity between SDDRE and SDRE, the counterpart of the SDDRE scheme to the FTHNOC problem could also be easily found in the literature.

The original theory of nonlinear optimal control dates from the 1960s, and over the decades since, various theoretical and practical aspects of the problem have been studied in the literature [9, 20, 35, 54, 134, 135, 149]. The long-established knowledge of nonlinear optimal control offers mature and well-documented techniques for solving the affine-in-control nonlinear optimization problem, based on the Bellman's dynamic programming (or calculus of variations). This approach reduces to solving a nonlinear 1st-order partial differential equation, known as the above-mentioned Hamilton-Jacobi-Bellman equation. The solution to the HJB equation gives the optimal performance index (or value/storage function), together with an optimal feedback control under smoothness assumptions. Alternatively, arising from the Pontryagin's minimum principle, the nonlinear optimal control problem can also be characterized locally in terms of the Hamiltonian dynamics, with respect to the classical calculus of variations.

Consider the continuous-time deterministic full-state-feedback ITHNOC problem, either for regulation or stabilization, where the system is autonomous, nonlinear in the state, and affine in the input, as below (for brevity, the obvious dependence on time t of some variables is omitted)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + B(\mathbf{x})\mathbf{u}, \ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1.1}$$

with the state vector $\mathbf{x} \in \mathbb{R}^n$, and unconstrained input vector $\mathbf{u}(t) \in \mathbb{R}^m$ $(1 \leq m \leq n)$ for each $t \in \mathbb{R}^+$. The vector fields $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $B_j(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^n$ are $\mathcal{C}^k$ mappings with $k \geq 0$ (at least continuous in $\mathbf{x}$), where $B_j(\mathbf{x})$ corresponds to the jth column of the matrix $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. Additionally, the minimization of an infinite-time performance index with a convex integrand (quadratic in $\mathbf{u}$ but nonquadratic in $\mathbf{x}$) is considered, as below

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \frac{1}{2}\int_{t_0}^{\infty} \left\{\mathbf{x}^T(t)Q(\mathbf{x})\mathbf{x}(t) + \mathbf{u}^T(t)R(\mathbf{x})\mathbf{u}(t)\right\} dt, \tag{1.2}$$

where $Q : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ (resp. $R : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}^{\mathrm{m}\times\mathrm{m}}$) is the state (resp. control/input) weighting matrix, and also state-dependent $\mathcal{C}^k$ matrix mapping with $k \geq 0$ (at least continuous in $\mathbf{x}$).

Define $\psi = \{\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^{\mathrm{m}} \mid \mathbf{u}(\cdot) \in L_2(\mathbb{R}^+)\}$ as the set of control functions, where $L_2(\mathbb{R}^+)$ is the Lebesque space, consisting of measurable square-integrable (vector-valued) functions $\mathbf{u} : \mathbb{R}^+ \to \mathbb{R}^{\mathrm{m}}$, such that $\int_{\mathbb{R}^+} ||\mathbf{u}||^2 dt < \infty$. Therefore $\mathbf{u}(\cdot) \in \psi$ is some appropriately bounded and measurable control scheme on $t \in \mathbb{R}^+$. Then, given $\mathbf{0} \in \Omega \subseteq \mathbb{R}^{\mathrm{n}}$ (resp. $\mathbf{x}_0 \in \Omega$), a bounded open set containing the origin (resp. an initial point), the ITHNOC problem on Ω is to minimize the performance index (1.2) regarding $\mathbf{u}(\cdot) \in U = \{\mathbf{u} \in \psi : \ \mathbf{x}(t) \in \Omega, \forall t \geq 0\}$, where U is the set of admissible controls such that the unique solution $\mathbf{x}(\cdot)|_{\mathbf{u}(\cdot)\in\psi}$ stays in Ω for all t, and approaches the origin as $t \to \infty$ for all $\mathbf{x}_0 \in \Omega$.

Under the above-specified conditions, a stabilizing feedback control as

$$\mathbf{u}(\mathbf{x}) = -\mathbf{k}(\mathbf{x}), \ \mathbf{k} \in \mathcal{C}^j(\Omega), \ j \geq 0, \ \mathbf{k}(\mathbf{0}) = \mathbf{0}, \tag{1.3}$$

which is then sought that will possibly/approximately minimize the performance index (1.2), subject to the considered system (1.1) while regulating/stabilizing the system to the origin for all $\mathbf{x} \in \Omega$, i.e. $\lim_{t\to\infty} \mathbf{x}(t) = 0$.

Remarkably, the ITHNOC problem on $\Omega \subseteq \mathbb{R}^{\mathrm{n}}$ is to minimize the performance index (1.2) for some $\mathbf{u}(\cdot) \in U$. And a solution to this problem is said to exist on Ω, if there exists a finite continuous nonnegative-definite value function (also known as the Lyapunov function) $V : \Omega \to \mathbb{R}^+$, as defined by

$$V(\mathbf{x}) := \inf_{\mathbf{u}(\cdot)\in U} J(\mathbf{x}, \mathbf{u}(\cdot)), \ \forall \mathbf{x} \in \Omega. \tag{1.4}$$

Ideally, V is a stationary solution to the Cauchy problem for the associated dynamic programming (Bellman's) equation, represented by the following first-order nonlinear HJB equation

$$\frac{\partial V(\mathbf{x})}{\partial t} + H(\mathbf{x}, \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}, \mathbf{u}) = 0, \tag{1.5}$$

where H is the Hamiltonian for the ITHNOC problem, i.e. the system (1.1) and the performance index (1.2), and given by

$$H = \inf_{\mathbf{u}(\cdot)\in U} \left\{ \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \left[\mathbf{f}(\mathbf{x}) + B(\mathbf{x})\mathbf{u}\right] + \frac{1}{2} \left[\mathbf{x}^T Q(\mathbf{x})\mathbf{x} + \mathbf{u}^T R(\mathbf{x})\mathbf{u}\right] \right\}. \tag{1.6}$$

Since the ITHNOC problem deals with the infinite-time horizon, V could be assumed stationary, i.e. $\frac{\partial V}{\partial t} = 0$, and then the HJB equation (1.5) becomes

$$\inf_{\mathbf{u}(\cdot)\in U} \left\{ \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \left[\mathbf{x}(\mathbf{x}) + B(\mathbf{x})\mathbf{u}\right] + \frac{1}{2}\left[\mathbf{x}^T Q(\mathbf{x})\mathbf{x} + \mathbf{u}^T R(\mathbf{x})\mathbf{u}\right] \right\},$$
$$\text{with the boundary condition } V(\mathbf{0}) = 0, \tag{1.7}$$

where the boundary condition comes from the requirement for the closed-loop stability, i.e. $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$. Finally, it is worth mentioning that

$$\left.\frac{\partial^2 H}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}_*} = R(\mathbf{x}), \tag{1.8}$$

where $\mathbf{u}_*$ denotes the optimal control of the ITHNOC problem. Therefore, if $R(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$ (the convexity condition), then the optimal control $\mathbf{u}_*$ that minimizes the HJB equation (1.7) must satisfy

$$\mathbf{0} = \left.\frac{\partial H^T}{\mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_*} = B^T(\mathbf{x})\frac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}} + R(\mathbf{x})\mathbf{u}_*, \text{ or equivalently}$$

$$\mathbf{u}_* = -R^{-1}(\mathbf{x})B^T(\mathbf{x})\frac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}}. \tag{1.9}$$

In general, it is not easy to obtain the optimal control (1.9) for the nonlinear systems, since the information of the value function $V(\mathbf{x})$ is not known *a priori*. However, in real-world applications, the systems encountered are most often in the nonlinear formulation, and thus practical engineers turn alternatively to a compromise.

Around 1970s, the linear system theory has been rigorously investigated and well established. Thus in view of the significance of practical engineering, linear systems have always been highly emphasized among the control community, since they are much easier to deal with mathematically and practically. The first step is to linearize the nonlinear dynamics around some nominal operating point, typically the origin, and through coordinate transformation, it is possible to relate any operating point to the origin. Certainly, such an approximation by linearization is only adequate over a limited domain around the operation point, and for large deviations, the effects are often investigated via simulations. In fact, such a guideline is widely adopted for practical control systems.

Among all the linear control techniques, the design via “Linear Quadratic Regulator (LQR)” receives great attention, and proves to be a successful strategy for numerous linear systems. Due to its simplicity in design and specific recovery of the optimal control, LQR design truly plays an important role in the progress of the control theory. Therefore, it is quite popular in most nonlinear control systems that the LQR design being applied through the linearization and, particularly to the ITHNOC problem. Such a strategy is often known as "Jacobian linearization", and described as follows.

At first, the Jacobian matrix (first-order Taylor-series expansion) around the origin is adopted, i.e. the original nonlinear dynamics (1.1) is approximated via its linear form (stationary dynamics with constant time-invariant matrices)

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{1.10}$$

where $A = \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}}$ and $B = B(\mathbf{0})$. Moreover, the performance index (1.2) is also "frozen" at the beginning, i.e. $R = R(\mathbf{0})$ and $Q = Q(\mathbf{0})$ with full rank decomposition as $Q = C^T C$. In this linear setting, it is well known that the optimal performance index (Lyapunov/value function) associated with the linear system (1.10), if it exists, must be of the form

$$V(\mathbf{x}) = J_*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T P\mathbf{x}, \tag{1.11}$$

for some unique constant $P = P^T \geq 0,\ P \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, which satisfies the following algebraic Riccati equation (ARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{1.12}$$

Note that, such an ARE (1.12) is just the reduced form of the HJB equation (1.7), with respect to the linear system (1.10). Moreover, the necessary and sufficient condition for the solvability of the ARE (1.12) is well-developed and could be found in the literature (e.g. [11, 24, 43, 79, 128, 146, 259–261]). Some of the results are summarized as the following Property 1.1.1

Property 1.1.1.

1. *ARE (1.12) admits a unique, symmetric, and stabilizing* $P \geq 0$
   $\Leftrightarrow$ *the pairs* (A, B) *is stabilizable, and* (A, C) *has no unobservable mode on the imaginary axis.*

2. *ARE (1.12) admits a unique, symmetric, and stabilizing* $P > 0$
   $\Leftrightarrow$ *the pairs* (A, B) *is stabilizable, and* (A, C) *has no unobservable mode on the imaginary axis and the left-half plane.*

Finally, the optimal feedback control regarding the linearized nonlinear dynamics (1.10) is thus obtained as

$$\mathbf{u}_* = -R^{-1}B^T P\mathbf{x}, \tag{1.13}$$

which is just the reduced form of the original nonlinear optimal control (1.9).

Ideally, the concept of the SDRE/SDDRE scheme builds on the Jacobian linearization design, but no linearization or truncation of nonlinearities are required. Loosely speaking, the SDRE/SDDRE scheme could be viewed as the *point-wise LQR design for nonlinear systems*, and that may explain why some different names for the SDRE scheme originate from, such as “Frozen Riccati Equation [116]”, “Apparent linearization [246]”, and “Extended linearization [92]”. In the following, the whole picture of the SDRE/SDDRE scheme is depicted, and substantial theoretical support will be provided in this thesis with simulation demonstrations via MATLAB®.

1.2 Problem Formulation

The SDDRE design for nonlinear systems can be described as follows, which is performed it pointwisely in $\mathbf{x}$ and in the finite-time horizon $t_0 \leq t \leq t_f$. Consider a class of nonlinear time-variant affine-in-input systems and a quadratic-like performance index as (1.14)-(1.15) below: (for brevity, the obvious dependence on time t of some variables is omitted)

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},t) + B(\mathbf{x},t)\mathbf{u} \tag{1.14}$$

$$\begin{aligned} J_{\text{SDDRE}} &= \frac{1}{2}\mathbf{x}^T(t_f)S(t)\mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \Big\{\mathbf{x}^T Q(\mathbf{x},t)\mathbf{x} \\ &\quad +\mathbf{u}^T R(\mathbf{x},t)\mathbf{u}\Big\}\, dt, \end{aligned} \tag{1.15}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^p$ denote the system states and control inputs, respectively, $\mathbf{f}(\mathbf{x},t) \in \mathbb{R}^n$, $B(\mathbf{x},t) \in \mathbb{R}^{n\times p}$, $S^T(t) = S(t) \geq 0$, $Q^T(\mathbf{x},t) = Q(\mathbf{x},t) \geq 0$, $R^T(\mathbf{x},t) = R(\mathbf{x},t) > 0$, and $(\cdot)^T$ denotes the transpose of a vector or a matrix. Conceptually, the procedure to obtain the SDDRE feedback control is briefly as the following three steps [98, 99, 111–113]:

1) Factorize $\mathbf{f}(\mathbf{x},t)$ into the appropriate SDC matrix representation as $\mathbf{f}(\mathbf{x},t) = A(\mathbf{x},t)\mathbf{x}$, where $A(\mathbf{x},t) \in \mathbb{R}^{n\times n}$.

2) If possible, solve it pointwisely (both in the state variable and time) the following SDDRE for $P(\mathbf{x}, t)$:

$$\begin{aligned}&A^T(\mathbf{x},t)P(\mathbf{x},t) + P(\mathbf{x},t)A(\mathbf{x},t) + Q(\mathbf{x},t)\\&-P(\mathbf{x},t)B(\mathbf{x},t)R^{-1}(\mathbf{x},t)B^T(\mathbf{x},t)P(\mathbf{x},t)\\&= -\dot{P}(\mathbf{x},t) \text{ and } P(\mathbf{x},t_f) = S(t),\end{aligned} \tag{1.16}$$

where $C(\mathbf{x}, t) \in \mathbb{R}^{q\times n}$ has full row rank and satisfies $Q(\mathbf{x}, t) = C(\mathbf{x}, t)^T C(\mathbf{x}, t)$, and $\dot{P}(\mathbf{x}, t)$ denotes the total time derivative of $P(\mathbf{x}, t)$ [111].

3) The control law is thus given by

$$\mathbf{u}_{\text{SDDRE}} = -K(\mathbf{x},t)\mathbf{x} = -R^{-1}(\mathbf{x},t)B^T(\mathbf{x},t)P(\mathbf{x},t)\mathbf{x}. \tag{1.17}$$

Similar to the SDDRE scheme, the SDRE strategy instead considers the infinite-time horizon counterpart ($t_f \to \infty$) with corresponding performance index as [51, 54]

$$J_{\text{SDRE}} = \frac{1}{2}\int_{t_0}^{\infty} \left\{\mathbf{x}^T Q(\mathbf{x},t)\mathbf{x} + \mathbf{u}^T R(\mathbf{x},t)\mathbf{u}\right\} dt \tag{1.18}$$

and solves it pointwisely the following SDRE for $P(\mathbf{x}, t)$:

$$\begin{aligned}&A^T(\mathbf{x},t)P(\mathbf{x},t) + P(\mathbf{x},t)A(\mathbf{x},t) + Q(\mathbf{x},t)\\&-P(\mathbf{x},t)B(\mathbf{x},t)R^{-1}(\mathbf{x},t)B^T(\mathbf{x},t)P(\mathbf{x},t) = 0\end{aligned} \tag{1.19}$$

; while the control law is given by

$$\mathbf{u}_{\text{SDRE}} = -K(\mathbf{x},t)\mathbf{x} = -R^{-1}(\mathbf{x},t)B^T(\mathbf{x},t)P(\mathbf{x},t)\mathbf{x}. \tag{1.20}$$

Obviously, both schemes highly rely on the feasible choices of SDC matrices for successful implementations, and are performed it point-wise in $\mathbf{x}$. For the SDRE scheme, the resulting closed-loop SDC matrix $A_{CL}(\mathbf{x}, t) := A(\mathbf{x}, t) - B(\mathbf{x}, t)R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t)$ is it point-wise Hurwitz everywhere; however, it does not imply global stability of the origin [56, 57, 104, 204, 234].

Note that, the two schemes are quite similar except mainly the time horizon considered, i.e. finite time horizon for the SDDRE scheme while infinite for the

SDRE scheme. Although the additional time dependency makes the SDDRE scheme more challenging and complicated than the SDRE scheme [112], in real-world applications, such a finite-horizon formulation is the only viable choice for many control problems, either fixed final time or final states [110]. For example, guidance control, landing of an airplane in a fixed downrange, or many trajectory-tracking/path-planning (optimization) problems. Note that, if the final time is not necessary to be fixed, but the final values of (monotonically changing) state variables are pre-specified and fixed, [109] formulates such a case into a finite-horizon problem through the change of variable, which the SDDRE scheme could be applicable to provide feedback controls.

1.3 Comparison with Similar Approaches

As described in previous two sections, the SDRE/SDDRE scheme originates from the reputed ***Jacobian linearization*** and ***LQR design***. Note that, in the SDRE design, there exists a design degree of freedom arising from the non-uniqueness of the SDC matrix, with infinitely many choices for general-order systems. In the ***Jacobian linearization*** design, the Jacobian matrix of the drift term is adopted, i.e. the first-order Taylor-series expansion around the origin (operating point). It is interesting to point out a connection between the two types of representations for the drift term, as the following lemma

Lemma 1.3.1. *[54, 112] For any choice of SDC factorizations* $\mathbf{f} = A(\mathbf{x}) \cdot \mathbf{x}$, $A(\mathbf{0})$ *is just the linearization of* $\mathbf{f}(\mathbf{x})$ *at the origin, i.e.* $A(\mathbf{0}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\big|_{\mathbf{x}=\mathbf{0}}$*, the Jacobian matrix of the drift term evaluated at the origin.*

Proof. In the literature, there are several different proofs. For example, [54] comes up with any two different SDC matrices, and justifies through algebraic arguments that they are of the same value when evaluated at the origin, together with the uniqueness of that value among all SDC matrices. On the other hand, [112] utilizes the mean value theorem and the concept of "vector of matrices [19]", the equivalence could also be proved as a result of the continuity of the SDC matrix. □

Although both ***Jacobian linearization*** and ***LQR design*** still maintain their popularity among practitioners for nonlinear control systems, with (approximated) linear control laws [54]. Those laws are only valid locally around the operating point, typically the origin. But, if the practical requirements specify that the system is to be operated under a wide range of conditions and, by applying techniques such as those two, then it may be necessary to obtain a set/sequence of approximate models (either by linearizing around different/subsequent operating points), and that is the idea of the reputed ***gain scheduling*** design [213, 220, 221]. Then, a sequence of scheduled controllers will be constructed, which are either activated successively as the system trajectories pass through conditions where the corresponding approximate models satisfy, or combined by continuously interpolating the point designs such that an overall controller for the nonlinear system is obtained [213]. It should be emphasized that, unlike ***gain scheduling*** and ***LQR design***, the SDRE/SDDRE scheme generates its control laws at every time instant, which typically depends on the sampling time for implementation, or the adopted time step for simulation. More specifically, in contrast to the ***gain scheduling***, no set/sequence of approximate models (and thus sequence of controls) are pre-determined. However, how to appropriately select an SDC matrix among all possibilities to meet some performance criterium *at each time instant* is definitely worth investigating, and in this thesis, we mainly focus on the full exploration of all possible SDC matrices (Section 1.5 describes an overview).

Indeed, from the practical viewpoint, the combined scheme of ***Jacobian linearization*** and ***gain scheduling*** presents a very effective solution to nonlinear control problems, without requiring severe structural assumptions on the plant model [213]. Additionally, this combined scheme also utilizes the well-established linear system theory, which is intuitively an appealing benefit in the formulation process. However, it is reported that ***gain scheduling*** could be a very tedious process, and consume enormous time and energy/effort, for wide variations in operating conditions [54]. Besides, for complex, high-order, or extremely nonlinear systems, since the fundamental nonlinear nature of the dynamics is not fully exploited (ignored by approximations), this may render the overall system's performance unsatisfactory and of limited value. But, for the SDRE/SDDRE scheme, no truncation of nonlinear terms are mandatory, and thus the complete information of the plant (regardless of the modeling error/uncertainty) is incorporated into the design.

Finally, the SDRE scheme is also similar to the reputed ***Model Predictive Control*** (***MPC***) with its nonlinear version called ***NMPC***, especially in the process industry [165, 167, 208, 209]. In both control schemes, based on the evaluation of the system's state at the current time step, a controller is designed via an optimization technique and implemented for the duration of the current time step. As a time step forward, the system's state is re-evaluated, then follows the optimization technique, and the operation repeats and continues [145]. On the other hand, the major differences between SDRE and (***N***)***MPC*** lie in the following elements:

1. During the design and analysis prior to the implementation, the SDRE scheme usually deals with the continuous-time domain, while the ***MPC***

considers the discrete-time counterpart and thus sampling/discretization is required.

2. The optimization operation of ***MPC*** is done in the predicted future interval, whose length is usually tens or hundreds of sampling time; But for SDRE, the design procedure is performed *point-wisely* at each time instant. For applications with fast and immediate response required, such as the interception of missiles, then the SDRE design seems to be more suitable. Other than that, currently ***MPC*** is more adopted due to its future predictions.

3. The optimization technique adopted in the SDRE scheme is from the LQR optimal control for linear systems, mainly solving the algebraic Riccati equation; while for the ***MPC***, generally the convex optimization technique [33] is utilized.

4. The ***MPC*** is considered to be repeatedly solving an open-loop problem [148], while the SDRE could be viewed as a closed-loop approach [145].

5. Regarding the estimation of "domain of attraction" (DOA) around the origin, for ***NMPC*** local stability of the origin could be guaranteed by invoking certain conditions (such as a zero state constraint at the end of the controller horizon) [164] and then, in order to determine the size of the DOA, the region in which feasible trajectories lie is acknowledged to be necessary, which generally could not be known *a priori*. On the other hand, Section 4.1 in this thesis tackles such a DOA-estimation problem via the SDRE scheme, based on various contributions [34, 85, 87, 88, 145], and results in a systematic procedure for *a priori* estimating the DOA.

1.4 State of the Art

The concept of the SDRE/SDDRE scheme was initially proposed by J. D. Pearson (Imperial College, London) in [203], which considers the FTHNOC for the nonlinear time-variant system. By it pointwisely "freezing" the system as linear and time-invariant, and then applying the developed LQR design, the main idea is to approximate the optimal control law. Years later, the SDRE scheme gained immense popularity in the control community, especially in the area of aerospace, aviation, and military-related industries. It is worth mentioning that the U.S. air force has devoted large efforts and highly expected its potentials, with representative scholars such as J. R. Cloutier, C. N. D'Souza, K. D. Hammett, C. P. Mracek, D. K. Parrish, D. B. Ridgely, and P. H. Zipfel.

Thanks to numerous contributions (e.g. Tables 1.1-1.2), the SDRE/SDDRE paradigm has been shown to be a promising and powerful control strategy from both the theoretical and practical viewpoints. Among them, T. Çimen (ROKETSAN Missiles Industries Inc., Turkey) summarized several survey papers to sum up the progress and status of the SDRE/SDDRE schemes, including his own findings (e.g. [48, 50, 52, 53, 55, 58, 59, 169]) and appearing in the *IFAC world congress* ([49], 2008), *Annual Reviews in Control* ([51], 2010), and *AIAA Journal of Guidance Control and Dynamics* ([54], 2012), and he is also the chair of the invited session for the SDRE paradigm during the last *IFAC world congress*.

Table 1.1: Some contributions to the SDRE/SDDRE paradigm published in several highly-regarded control journals.

Theoretical Developments	
Automatica	[104, 153, 246, 254, 255]
IEEE Trans. on Automatic Control	[37, 72, 194, 222]
AIAA J. Guid. Control Dyn.	[34, 101, 111, 230, 233, 233]
Trans. ASME, J. Dyn. Syst. Meas. Control	[94, 145]
Systems & Control Letters	[55, 166]
Practical Impact	
IEEE Trans. on Control Systems Technology	[88, 113, 191, 210, 243]
AIAA J. Guid. Control Dyn.	[29, 106]
IEEE Trans. on Industrial/Power Electronics	[76, 77, 226]
IEEE Trans. on Industrial Informatics	[151]
Survey Papers	
Annual Reviews in Control	[51] (2010)
AIAA J. Guid. Control Dyn.	[54] (2012)

The rest of this section provides literature surveys on the developments of the SDRE/SDDRE paradigm. Section 1.4.1 includes the theoretical developments; while Section 1.4.2 discusses the (successful) practical applications, as summarized in Table 1.2.

1.4.1 Theoretical Developments

DOA Analysis

[34] Based on the Lyapunov analysis, Bracci et al. proposed a less conservative procedure for estimating a higher bound of the DOA, as compared to the

Table 1.2: Developments of the SDRE/SDDRE paradigm from both the theoretical and practical viewpoint (alphabetical order).

Theoretical Developments (Sec. 1.4.1)	
DOA Analysis	[34, 41, 87, 145, 166, 218, 246]
DOF of the SDC	[62, 101, 152, 153]
(Global) Asymptotic Stability	[19, 88, 100, 101, 143, 145]
Methodical Interaction	[94, 139–142, 144]
Optimality and Suboptimality	[178, 206, 222]
Robustness	[62, 70–72]
Practical Impact (Sec. 1.4.2)	
Aerospace and Aviation	[62, 143, 178, 235, 236]
AUV	[125, 182, 192, 193, 232, 252]
Benchmark Problems	[113, 178]
Biomedical and Biological Applications	[16, 17, 122, 191, 202]
Chemical Engineering	[44, 163]
(Electric) Vehicles' Design	[8, 129, 151, 155, 243]
Permanent-Magnet Synchronous Motor	[76, 77]
UAV	[27–30, 96, 185, 210, 226]

existing techniques at the time (e.g. [87]).

[41] Motivated by the contraction analysis, Chang and Chung originated a method to estimate DOA. Through examples, the superiority of the proposed scheme is demonstrated as compared to some relevant works (e.g. [34]).

[87, 166] McCaffrey and Banks presented an analysis for estimating the large scale DOA, by the geometrical construction of a viscosity-type Lyapunov function from a stable Lagrangian manifold. On the other hand, apart from some brute-force time-domain simulations, Erdem and Alleyne proposed the analytical study of DOA via the SDRE scheme, based on defining an overvaluing comparison system [32] for the original one and using the vector norms [31, 211]. It is worth mentioning that, they initiated the interest in the DOA analysis via SDRE among the control community (e.g. [34, 41]).

[145] Under the main assumptions of it point-wise availability of full state measurement vector and stabilizability, Langson and Alleyne presented sufficient conditions to estimate the DOA, and the corresponding convergence behavior is also studied.

[218] By the sum of squares polynomials, P. Seiler (UIUC) transformed the DOA estimation into a semidefinite programming problem [198, 199], and in the simulation setup, the proposed scheme seem to obtain larger DOA than [87].

[246] Conditions on the local asymptotical stability of the resulting closed-loop system are presented. Different to other contributions, they deal with more general nonlinear nonautonomous systems.

DOF of the SDC

[62] (1) For the general-order system, the drift term has infinitely many factorizations, which could be viewed as a design DOF and used to enhance the system performance.

(2) Demonstrated via examples, that the tuning of the weighting matrices is possible to avoid exceeding the prescribed limits on the system states and inputs.

[101] Hammett et al. studies the relation and comparison between "factored controllability" and "true controllability" with respect to the factorization of the SDC matrix, resulting conditions on the local (resp. global) equivalence between the two types of controllability, and demonstrated by a fifth-order dual-spin spacecraft. Note that more detailed results could be found in his Ph. D. thesis [100].

(Global) Asymptotic Stability

[19, 145] The authors tried to provide certain conditions to guarantee GAS for the considered class of nonlinear systems, and it seems that further investigations are required [179, 180, 234].

[62] (1) For the scalar system with state-dependent weighting matrices $Q(\mathbf{x})$ and $R(\mathbf{x})$, the origin of the closed-loop system via the SDRE scheme is GAS.

(2) If $A_{CL}(\mathbf{x})$ is symmetric, then the origin under the SDRE scheme is GAS.

[70–72] Curtis and Beard pointed out that, the proposed satisficing technique could be equipped with the SDRE scheme to retain the essential behavior, while ensuring closed-loop asymptotic stability.

[88] Erdem and Alleyne (UIUC) advanced a novel SDC factorization for the second-order nonlinear system, which satisfies sufficient conditions for GAS.

[143] This survey paper provides a summary of stability analysis on the SDRE scheme and, most importantly, to justify the equivalent importance between practical rules of thumb and theoretical stability proofs, with respect to real-world applications.

[222] In cooperation with J. S. Shamma (UCLA and presently Georgia Tech.), Cloutier investigates the diversity of the SDC factorizations, and concluded that the origin could be asymptotically stabilized by a specific SDC matrix, if (1) the original nonlinear plant could actually be stabilized and; (2) there exists a Lyapunov function with star-convex level sets.

Methodical Interaction - Adaptive Control Design

[141] Lam et al. presents an adaptive controller solution using the SDRE scheme to augment the flight control system, to address the operating environment of the dynamic interface.

[142, 144] Perceived as an equivalent approach to the indirect adaptive control paradigm, the SDRE scheme could re-compute the controller gain in real-time based on the SDC matrix, which captures new knowledge of the plant dynamics from the dynamic state vector signatures without explicitly relying on an onboard parameter estimator. In the application of a reusable launch vehicle during reentry conditions [142], the SDRE scheme adopted is viewed as an adaptive guidance and control design to perform rate stabilization, subject to four drastic flight conditions.

Methodical Interaction - Chaotic/Uncertain Systems Applications

[139] Aimed at the chaotic systems with disturbances, Y.-L. Kuo transformed the original system into an optimal control problem and then solved by the SDRE scheme. Via simulations, the proposed scheme could regulate the error dynamics of the chaos synchronization.

[140] Y.-L. Kuo applied the combined SDRE-ISMC scheme (e.g. [158]) to a chaotic system, such that the DOF from the SDRE method is inherited while the steady-state error could be minimized by the ISMC design.

[151, 155] Applied the combined SDRE-ISMC scheme, Liang et al. explored the design of active reliable control for a class of uncertain nonlinear affine systems, showing that the main advantages (e.g. robustness, rapid response, and easy implementation) of the ISMC scheme are maintained. Finally, the proposed scheme is demonstrated via a vehicle brake control (antilock brake system).

Methodical Interaction - Singularly Perturbed Systems

[94] Ghadami et el. combined the SDRE scheme and the singular perturbation theory, harvesting mainly the advantages of no requirement in knowing the Jacobian matrix (thus the simplicity of LQR method is inherited) and reduction of the original system into low-order subsystems, respectively. Note that the proposed scheme is the first contribution for the considered class of nonlinear singularly perturbed systems.

Optimality and Suboptimality

[37] J. H. Burghart uses the Taylor series expansion for the it point-wise feedback gain matrix, to analyze the suboptimal control via the SDRE scheme. The presented method is simple, easily implemented, and without using common iterative techniques or any true optimal solutions.

[62] (1) For the scalar system with state-dependent weighting matrices $Q(\mathbf{x})$ and $R(\mathbf{x})$, the SDRE scheme satisfies all necessary conditions of the corresponding optimal control law.

(2) The SDRE scheme will asymptotically satisfy the costate function of the corresponding optimal control strategy, in a quadratic convergence rate.

[246] Wernli and Cook (Lockheed Martin) extended the results of [203] to a rather general nonlinear time-variant system, concluding that: (1) conditions on the existence of the SDRE suboptimal control law of the resulting closed-loop system; (2) usage of Taylor series to approximate the optimal control law with an algorithm, and the increased possibility of real implementation (since the computational burden could be significantly reduced).

[254] Yoshida and Loparo (IBM and Case Western Reserve Univ.) exploits the well-documented LQR technique for the analytic nonlinear systems, investigating both the FTHNOC and ITHNOC problems. As many other contributions, the assumption of controllability and observability for the adopted SDC matrix (the Jacobian matrix of the drift term evaluated at the origin) is required.

Robustness

[62] By extending to the nonlinear H_∞ control [239, 240], the presented SDRE H_∞ approach locally satisfies certain conditions (i.e., the L_2-gain boundedness and the closed-loop system being internally stable [121]), such that the controlled system is robust to a certain extent.

[70–72] By it pointwisely projecting the SDRE controller onto the robust satisficing set, the robustness of the closed-loop system is guaranteed.

1.4.2 Practical Impact

Aerospace and Aviation

[100, 101] Angular momentum control of an axial dual-spin spacecraft, exhibiting highly nonlinear dynamics and limited controllability.

[143] Designs of an SAV control, a helicopter flight control, and a satellite pointing control.

[255] Scholars in Caltech (Control and Dynamical Systems program) applied the SDRE scheme for the vector thrust control using the Caltech ducted fan, and compared the performance with various nonlinear control strategies such as model predictive control and LPV methods; however, in the simulation setup, the performance of the SDRE scheme seems to be worse than that of the other methods.

[235, 236] In the simulation setup [236], the authors demonstrate the application of SDRE missile guidance, which outperforms the conventional PPN guidance regarding time cost but not for the energy cost. Moreover, in another setup [235], the proposed AIPN guidance law outperforms the SDRE based strategy with respect to the considered performance index.

AUV plays an ideal alternative to humans in making decisions and take control actions more accurately and reliably without human intervention, especially during long-term operations.

[125] Aiming for homing and docking tasks, Jantapremjit and Wilson proposed an high-order sliding mode control via the SDRE scheme, with robustness and elimination of the chattering effect.

[182] Naik and Singh treated the dive plane control problem of AUV via the SDRE scheme, which retains the essential nonlinearities and proves to be effective for both the minimum and nonminimum phase AUV models.

[192, 193] The robust depth control problem was considered, Pan and Xin showed that the proposed indirect robust control drives the AUV to the desired depth precisely, with smoother transient responses and smaller control efforts, as compared with the SDRE scheme.

Benchmark Problems

[113] Heydari et al. incorporated the SDDRE scheme to formulate and solve the process planning problem, such that the machining time in turning operations could be minimized.

[178] Applied to the renowned benchmark problem [36], Mracek and Cloutier found out that the SDRE scheme almost recovers the open-loop optimal control law, if neglecting disturbances and errors. Moreover, through simulation, the closed-loop system is robust against parametric variations and attenuates sinusoidal disturbances.

Biomedical and Biological Applications : animal population maintenance, HIV and CVD problems, cancer and diabetes mellitus treatments.

[16, 17] "HIV and CVD Problems": The authors (Stanford, MIT, North Carolina State Univ., and etc.) highly credits the SDRE scheme, with respect to their plentiful successful applications on the regulation of the growth of thin films in a high pressure CVD, and the development of optimal dynamic multi-drug therapies for HIV infection.

[122] "Cancer Treatment": Based on a nonlinear tumor growth model with normal tissue, tumor and immune cells, Itik et al. treats the chemotherapy administration as a control input and the amount of administered drug is determined via the SDRE scheme. As simulations suggest, the proposed scheme not only drives the amount of tumor cells to the healthy equilibrium but also minimizes the drug used. Currently, the authors are dealing with a quantitative cancer model with real clinical parameters, and the corresponding problem such as drug scheduling.

[191] "Animal Population Maintenance": Padhi and Balakrishnan incorporated the SDRE scheme into the proposed technique, specifically to generate the snap-shot solutions and for pretraining the networks, to

manage the beaver population such that the caused nuisances (e.g. flooding, canals-blocking and timbers-cutting) could be avoided.

[202] "Treatment of Diabetes Mellitus": Acted as a preliminary and theoretical study into a control methodology for an artificial pancreas, Parrish and Ridgely applied the SDRE paradigm effectively and successfully to regulate the amount of insulin delivered to a patient based on measuring this blood glucose level, with respect to the nonlinear model [183]. It is emphasized that more significant efforts and advances have to be made before any clinical trials.

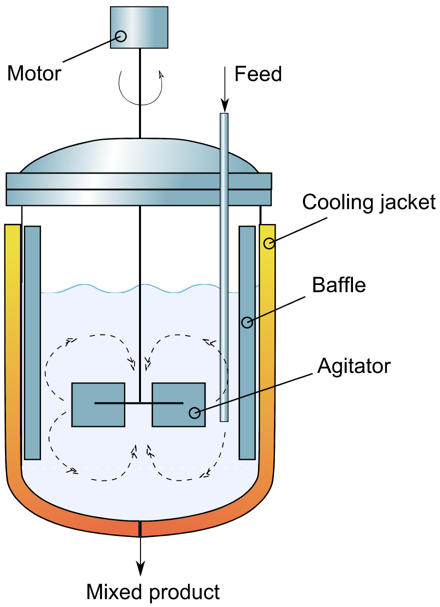

Figure 1.1: Diagram of a continuous stirred tank reactor [1].

Chemical Engineering

[44] Chen et al. (UCLA and Univ. of Texas, Austin) adopted the Jacobian matrix for the SDC in the SDRE scheme, to control the unstable steady state of a nonisothermal continuous stirred tank reactor, as in Fig. 1.1. Some issues related to the presented dynamic model like stability, optimality (via power series approximations), and constraint satisfaction regions are also identified.

[163] Aiming at both the constrained and unconstrained ITHNOC problems, Manousiouthakis and Chmielewski (UCLA) proposed an SDRE based approach to evaluate the cost function of the unstrained ITHNOC problem, addressing issues like stability and optimality of the resulting ITHNOC policy. Such results are then employed to the constrained case, and several properties are also established. For demonstration, the reactor control are adopted for both cases.

(Electric) Vehicles' Design

[8] Based on a nonlinear vehicle model with nonlinear tire characteristics, an optimal Vehicle Dynamic Control strategy via the SDRE scheme is developed and experimentally evaluated on a Jaguar XF test vehicle, which is capable of stabilizing the vehicle with less effect on the longitudinal motion, by scholars in TNO, Netherlands.

[129] Researchers in TNO advanced an SDRE controller which maximizes the regenerative braking energy of a real electric vehicle, equipped with a central electric motor on its front axle. The originality comes from the idea in deriving the SDC matrix of the vehicle system, based on the output of a vehicle state estimator and the magic formula tyre model (e.g. [190]), which also provides an appropriate future framework to include more actuators (such as the active suspension and active steering).

[243] Villagra et al. (PSA Peugeot Citroën) presented a sensitivity-based methodology to choose the best possible gains parameterization via SDRE for the vehicle steering control.

Permanent-Magnet Synchronous Motor (PMSM)

[76] In this practical contribution by Do et al., both the optimal speed controller and near optimal load torque observer for the PMSM are designed based on the SDRE scheme, with the stability analytically proven. From both the simulation and experimental results (tested using TMS320F28335 DSP), it is shown that the feasible SDRE-based design can ensure better performance (e.g. no overshoot and fast transient response in speed tracking), than conventional LQR and PI controllers.

[77] In addition to [76], the authors deal with the IPMSM, and the asymptotic stabilities are guaranteed for both the proposed controller and observer. Verified through experiments, such observer-based suboptimal control via

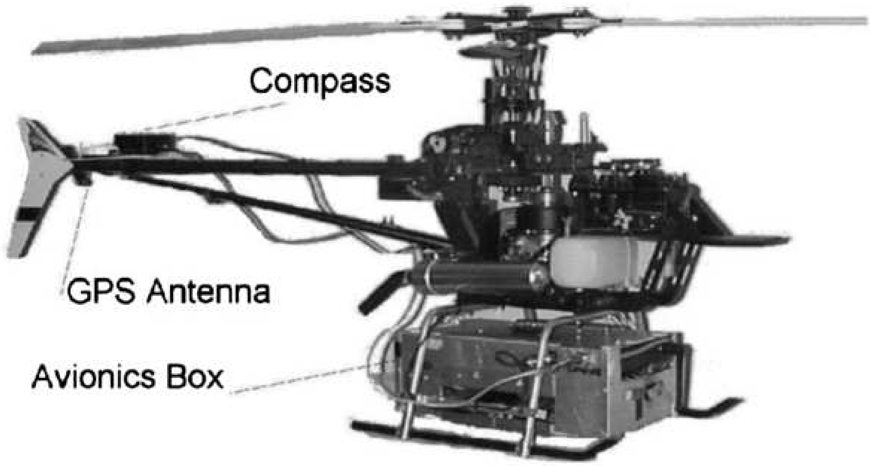

Figure 1.2: XCell-90 helicopter with sensors and avionics box [29, 93].

the SDRE scheme is easy-to-implement, because the solution of the SDRE could be Taylor-series approximated offline, and ensures faster dynamic response, smaller steady-state error, and more robust than LQR and PI controllers.

UAV

[27–30] Experimentally tested on both the MIT research vehicle XCell-90 in Fig. 1.2 and Georgia Tech. GTMAX (based on YAMAHA R-Max) in Fig. 1.3, Bogdanov et al. successfully realized the SDRE scheme in real-time applications of twelve system states, which computes the control in approximately 14 ms (about 70 Hz) using the 300 MHz Geode GX1 microprocessor. Although these are rather small-scale helicopters, it is emphasized that the proposed scheme could be easily generalized to more complex models. Note that, an easy construction for a feasible SDC matrix is presented in Section 2.3, which could alleviate the design burden in the initial stage of this application.

[210] Ren and Beard (Young Brigham Univ.) considered the trajectory control for UAV, with one of the controllers based on the SDRE scheme. At the expense of large velocity and heading rate commands, the SDRE controller could result in much better performance if input constraints are neglected.

Figure 1.3: GTMAX helicopter (YAMAHA R-Max) [29, 126].

Besides the above contributions, it is worth mentioning that in [189], the authors investigate the optimal tracking problem with the main results formulated in Theorems 1-2. However in both proofs the authors seem to consider only the case of $P(t)$ dependent on time but not on the state, while the more general and common case of $P(\mathbf{x}, t)$ is not addressed. Readers can easily infer this restriction from the lines below Eqs. (42) and (49). Unfortunately, the authors have nor mentioned explicitly this important restriction $P(t)$ and nor $P(\mathbf{x}, t)$ in their paper. Secondly, the assumption of controllable and observable state-dependent coefficient matrix could be removed, with respect to the findings in [153]. Finally, the problem formulation of state-dependent (differential) Riccati equation technique seems to differ from the consensus in the literature (e.g. [49, 51, 54]).

Last but not least, other contributions devoted to the SDRE/SDDRE scheme are not addressed but important as well, such as the "Estimator (Filter) Design" [23,62,97,124,175,196,214,215], which mainly deals with the observability. Such design is not addressed in detail because of the duality of controllability and observability, and the above survey includes mostly control problems.

1.5 Thesis Overview

In spite of all the theoretical developments and successful real-world applications as summarized in Section 1.4, in the initial stage of the SDRE/SDDRE scheme, how to systematically construct feasible SDC matrices to ensure the solvability of the corresponding SDRE/SDDRE is still under development, especially when the system dynamics are complicated or when the SDRE solvability condition is violated. For example, the UAV dynamics [29] exhibit twelve system state variables, which does require some design efforts to symbolically formulate a feasible SDC matrix. In addition, most of the contributions require an assumption on the feasible SDC matrix to ensure the solvability of the corresponding SDRE/SDDRE, and such a requirement still exists in some recent works [18, 40, 50, 94, 113, 133, 139, 194, 233, 253]. *In this thesis, the relations between the SDC matrices and the corresponding solutions' properties, namely existence, uniqueness, positive semi-definiteness, and positive definiteness, of SDRE (resp., SDDRE) are investigated. As s step forward, we will replace the above-mentioned requirement by a necessary and sufficient condition, together with a easy construction of a feasible SDC matrix and a representation of all feasible SDC matrices.* Specifically, the following three problems (Problems 1-3) for general-order nonlinear time-variant systems are investigated and solved, together with a formulation of feasible and implementable solutions.

Problem 1. Given an SDC matrix, how to efficiently determine the properties (i.e., existence, uniqueness, positive semi-definiteness, and positive definiteness) of solution of the corresponding SDRE/SDDRE?

Problem 2. When the SDRE/SDDRE solvability condition is satisfied, how to easily construct an appropriate SDC matrix with specific properties (i.e., existence, uniqueness, positive semi-definiteness, and positive definiteness) of the corresponding solution?

Problem 3. In connection with Problem 2, how to parameterize all the feasible SDC matrices with specific properties (i.e., existence, uniqueness, positive semi-definiteness, and positive definiteness) of the corresponding solution?

In addition, we provide more theoretical results and establish connections to existing contributions with respect to the SDRE/SDDRE control strategy.

To continue, the following notations are adopted and, for ease of reading, are tabularized at page.ix. Let $W \in \mathbb{R}^{p\times n}$ with $p < n$ and $\mathrm{rank}(W) = p$. We define $W^{\perp} = N(W)$, null space of W, and $W_{\perp} \in \mathbb{R}^{n\times(n-p)}$ as a selected constant matrix having orthonormal columns and satisfying $WW_{\perp} = 0$. Clearly, $W^{\perp}$ is a vector space of dimension $n - p$, and the column vectors of $W_{\perp}$ form an orthonormal basis of $W^{\perp}$. Similarly, if $W \in \mathbb{R}^{n\times q}$ and $\mathrm{rank}(W) = q < n$, we define $W^{\perp} = \{\mathbf{w}^T \mid \mathbf{w} \in N(W^T)\}$ and $W_{\perp} \in \mathbb{R}^{(n-q)\times n}$ as a selected constant matrix having orthonormal rows and satisfying $W_{\perp}W = 0$. Additionally, we denote $\mathbb{R}^{n*} = \{\mathbf{x}^{\mathrm{T}} | \mathbf{x} \in \mathbb{R}^n\}$, known as the dual space of $\mathbb{R}^n$, and $\mathbb{R}^{-}$ as the set of negative real numbers.

Unless otherwise mentioned in this thesis, we denote for brevity $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$, $A = A(\mathbf{x}, t)$, $B = B(\mathbf{x}, t)$, $C = C(\mathbf{x}, t)$, $Q = Q(\mathbf{x}, t)$, $R = R(\mathbf{x}, t)$, $P = P(\mathbf{x}, t)$, $S = S(t)$, $A_{CL} = A_{CL}(\mathbf{x}, t)$, and assume that, without loss of any generality, B (resp., C) has full column (resp., row) rank. In addition, denote the following SDC matrix sets $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$, $\mathcal{A}^d$, $\mathcal{A}^l$, and $\mathcal{A}^i$ as the sets of A such that $A\mathbf{x} = \mathbf{f}$, (A, B) is controllable, (A, B) is stabilizable, (A, C) is observable, (A, C) is detectable, (A, C) has no unobservable mode on the left-half plane (LHP) and $j\omega$-axis, and (A, C) has no unobservable mode on the $j\omega$-axis, respectively. Moreover, $\mathcal{A}_{\mathbf{xf}}^{\alpha\beta} = \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^\alpha \cap \mathcal{A}^\beta$, $\alpha = c, s$, $\beta = o, d, l, i$. Finally, let $A_p = arg \min_{A \in \mathcal{A}_{\mathbf{xf}}} ||A||_F = \frac{1}{||\mathbf{x}||^2} \mathbf{f}\mathbf{x}^T$, where $|| \cdot ||_F$ is the Frobenius norm, and then the SDC matrix set $\mathcal{A}_{\mathbf{xf}}$ becomes

$$\mathcal{A}_{\mathbf{xf}} = \left\{ A_p + K\mathbf{x}_\perp \mid K \in \mathbb{R}^{n \times (n-1)} \right\} \subset \mathbb{R}^{n \times n}. \tag{1.21}$$

The text is organized as follows.

- **Chapter 2** serves as a preliminary stage for the derivation afterwards. In this chapter, at first the necessary background knowledge regarding the algebraic Riccati equation, the linear system theory, and the LQR design are recalled, and then by it pointwisely "freezing" the system dynamics at each time instant, the above-mentioned results with respect to the LTI system could be applied, and the relation between the SDC matrices and the corresponding solutions' properties (namely existence, uniqueness, positive semi-definiteness, and positive definiteness) of SDRE/SDDRE (1.19/1.16) is investigated in-depth. Therefore, based on the derivations in this chapter, Problem 2 could be solved.
- **Chapter 3** continues the study in Chapter 2. As a step forward to the previous chapter, several required mathematical preliminaries from the developed linear system theory are reminded first, and then a possible way to parameterize/represent all feasible SDC matrices will be discussed, which would correspond to a specific solutions' property (namely existence, uniqueness, positive semi-definiteness, and positive definiteness) of SDRE/SDDRE (1.19/1.16). Hence, Problem 3 could be solved, and some of the derivations could be utilized to solve Problem 1.
- **Chapter 4** connects the proposed scheme to the literature. In recent years, there are more and more scholars worldwide devoting efforts in the SDRE/SDDRE scheme, some of the interesting topics like domain/basin of attraction, computational performance of solving SDRE (1.19), tracking or command following problem, closed-form solution of SDDRE (1.16), and the ultimate optimality issue. Therefore, by

connecting the proposed results to these topics, more theoretical support would be provided for the SDRE/SDDRE scheme, such that the numerous applications could be brought to a higher level.

- **Chapter 5** demonstrates the effectiveness of the proposed scheme. From both the illustrative viewpoint via textbooks' examples and the real-world applications (satellite attitude control and vector thrust control), we demonstrate the proposed construction of feasible SDC matrices, such that the SDRE scheme could be successfully continued at those states where the associated SDRE is unsolvable, yet the presented existence conditions hold there. Note that the system states corresponding to an unsolvable SDRE (1.19) may constitute points, lines, regions in the phase plane, or even throughout the whole considered time horizon. On the other hand, harvesting from the recent findings in the differential Riccati equation, the proposed SDDRE scheme with closed-form solution is also successfully demonstrated via the considered simulation setup.

- **Chapter 6** gives closing and summarizing remarks, which emphasize the importance of the considered problems, the relevance to the literature, and thus the contribution of this thesis. In addition, several interesting directions for further research are also suggested, to keep the SDRE/SDDRE scheme progressing promisingly.

Chapter 2

Existence Conditions for Feasible SDC Matrices [1]

Before we derive the solution to Problems 1-3, first we need to investigate more in-depth into the relation between the SDC matrices and the corresponding solutions' properties, namely existence, uniqueness, positive semi-definiteness, and positive definiteness, of SDRE (resp., SDDRE).

For example, it is well known [138, 261] that a unique positive semi-definite solution P in (1.19) exists, rendering the closed-loop SDC matrix A_{CL} it pointwise Hurwitz, if and only if both the conditions "(A, B) is stabilizable" and "(A, C) has no unobservable mode on the $j\omega$-axis" are satisfied. Obviously, such symbolic checking condition is generally not easy to implement, especially when the system dynamics are complicated. Moreover, several authors have provided various guidelines on how to systematically construct SDC matrices [51, 62]; however, there is no guideline on the construction of SDC matrices when the SDRE solvability condition is violated, which may result in the SDRE scheme being terminated. For instance, consider the following system

Example 1.

$$\dot{x}_1 = -x_2 \quad \text{and} \quad \dot{x}_2 = x_1 + x_2 u, \tag{2.1}$$

with $Q(\mathbf{x}) = I_2$ and $R(\mathbf{x}) = 1$. Clearly, $\mathbf{f}(\mathbf{x}) = [-x_2, x_1]^T$ and $B(\mathbf{x}) = [0, x_2]^T$. Suppose that an SDC matrix representation is given as $a_{11}(\mathbf{x}) = a_{22}(\mathbf{x}) = 0$, $a_{12}(\mathbf{x}) = -1$ and $a_{21}(\mathbf{x}) = 1$, where $a_{ij}(\mathbf{x})$ denotes the (i, j)-entry of the

[1] Journal and conference versions at [152, 153]

matrix $A(\mathbf{x})$. Then, *point-wisely* at every system state, $(A(\mathbf{x}), C(\mathbf{x}))$ is always observable, but $(A(\mathbf{x}), B(\mathbf{x}))$ is not stabilizable at the nonzero states where $x_2 = 0$.

By direct calculation, the SDRE given by (1.19) does not have any positive semi-definite solution $P(\mathbf{x})$ when $x_2 = 0$, in which case the SDRE scheme will fail to operate. However, it will become clear later in this chapter (see Theorem 2.2.1), at those nonzero states $\mathbf{x}$ of $x_2 = 0$, there always exists a feasible SDC matrix representation that makes the SDRE (1.19) solvable and the resulting $A_{CL}(\mathbf{x})$ matrix a Hurwitz matrix.

2.1 Preliminary Results

It will be shown in Section 2.2 that, once $\mathcal{A}^{so}_{\mathbf{xf}} \neq \emptyset$ (note that $\mathcal{A}^{so}_{\mathbf{xf}} \subseteq \mathcal{A}^{si}_{\mathbf{xf}} \subseteq \mathcal{A}^{sl}_{\mathbf{xf}}$), there always exists a real diagonalizable SDC matrix $A \in \mathcal{A}^{so}_{\mathbf{xf}}$ with real eigenvalues. Therefore, we consider a real matrix A in the form of

$$A = MDM^{-1} \tag{2.2}$$

where $D = \text{diag}[\lambda_1, \cdots, \lambda_n] \in \mathbb{R}^{n\times n}$, $M = [\mathbf{p}_1, \cdots, \mathbf{p}_n] \in \mathbb{R}^{n\times n}$ is nonsingular and $M^{-1} = [\mathbf{q}_1, \cdots, \mathbf{q}_n]^T \in \mathbb{R}^{n\times n}$. Clearly, $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A, and $\mathbf{p}_i$ and $\mathbf{q}_i^T$ are the right and the left eigenvectors of A associated with λ_i, respectively. We have the next result and the proof of which is an simple application of the Popov-Belevitch-Hautus (PBH) test [43]:

Lemma 2.1.1. *Let A be factorized in the form of (2.2). Then*

(i) $A\mathbf{x} = \mathbf{f} \Leftrightarrow \forall i, \lambda_i \mathbf{q}_i^T \mathbf{x} = \mathbf{q}_i^T \mathbf{f} \Leftrightarrow \forall i, \mathbf{q}_i^T(\lambda_i \mathbf{x} - \mathbf{f}) = 0.$

(ii) *(A, C) is observable (resp., detectable) $\Leftrightarrow \forall i,\ \mathbf{p}_i \notin C^\perp$ (resp., $\mathbf{p}_i \notin C^\perp$ whenever $\lambda_i \geq 0$). Moreover, (A, C) has no unobservable mode on the $j\omega$-axis $\Leftrightarrow \mathbf{p}_i \notin C^\perp$ for $\lambda_i = 0$.*

(iii) *(A, B) is controllable (resp., stabilizable) $\Leftrightarrow \forall i,\ \mathbf{q}_i^T \notin B^\perp$ (resp., $\mathbf{q}_i^T \notin B^\perp$ whenever $\lambda_i \geq 0$).*

Proof. (i) The result follows from writing $A\mathbf{x} = \mathbf{f}$ in the form of $DM^{-1}\mathbf{x} = M^{-1}\mathbf{f}$ and subsequently comparing both sides componentwise.

(ii) From the PBH test [43], (A, C) is unobservable $\Leftrightarrow \exists$ an eigenpair $(\lambda_i, \mathbf{p}_i)$ such that $C\mathbf{p}_i = \mathbf{0}$, i.e., $\mathbf{p}_i \in C^\perp$. Thus, $\text{rank}\left(\begin{pmatrix} C \\ \lambda_i I - A \end{pmatrix}\right) = n$, i.e.,

$\begin{pmatrix} C \\ \lambda_i I - A \end{pmatrix} \mathbf{p} \neq \mathbf{0}$ for any $\mathbf{p} \neq \mathbf{0}$. It is clear that $(\lambda_i I - A)\mathbf{p} = \mathbf{0} \Leftrightarrow (\lambda_i, \mathbf{p})$ is an eigenpair of A or $\mathbf{p} = \mathbf{0}$. It follows that (A, C) is observable $\Leftrightarrow \forall i, \mathbf{p}_i \notin C^{\perp}$. The cases for detectability and no unobservable mode on the $j\omega$-axis of (A, C) follow readily by replacing "$\forall i$" with "whenever $\lambda_i \geq 0$" and "whenever $\lambda_i = 0$," respectively.

(iii) Follows from (ii) and the duality property of controllability and observability.

It is known that (A, B) is controllable if and only if (A^T, B^T) is observable [43]. Since $(\lambda_i, \mathbf{q}_i), i = 1, \cdots, n$, are eigenpairs of A^T, we have from the proof of (ii) that (A^T, B^T) is observable if and only if $B^T \mathbf{q}_i \neq 0$, i.e., $\mathbf{q}_i^T B \neq \mathbf{0}$, for all i. Thus, (A, B) is controllable if and only if $\mathbf{q}_i^T \notin B^{\perp}$ for all i. The case for stabilizability of (A, B) can also be derived easily. □

In addition, we also need the following three results:

Lemma 2.1.2. *Let $\mathcal{V}$ be a k-dimensional vector subspace of $\mathrm{I\!R}^{n*}$, where $k < n$, and $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$ are linearly independent (LI) vectors with $\mathbf{q}_1^T \notin \mathcal{V}$. Then there exists $\mathbf{q}_{k+1}^T \in \mathcal{V}$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{k+1}^T\}$ are LI.*

Proof. Suppose that such $\mathbf{q}_{k+1}^T$ does not exist. Then $\mathcal{V} \subset \mathrm{span}\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$. Since both $\mathcal{V}$ and $\mathrm{span}\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$ have dimension k, we have $\mathcal{V} = \mathrm{span}\{\mathbf{q}_1^T, \cdots, \mathbf{q}_k^T\}$, and thus $\mathbf{q}_1^T \in \mathcal{V}$, a contradiction. This completes the proof. □

Lemma 2.1.3. *Let $\mathcal{V}$ be a $(n-1)$-dimensional subspace of $\mathrm{I\!R}^{n*}$ and $W := \{\mathbf{v}_1^T, \cdots, \mathbf{v}_n^T\}$ be a basis of $\mathrm{I\!R}^{n*}$ with $\mathbf{v}_i^T \notin \mathcal{V}$ for all i. Define $\mathcal{W}_i := \mathrm{span}\{W \backslash \{\mathbf{v}_i^T\}\}$ for $1 \leq i \leq n$. Then $\mathcal{V} \not\subset \cup_{i=1}^n \mathcal{W}_i$ and there exists a nonzero $\mathbf{v}^T \in \mathcal{V}$ such that $\mathbf{v}^T = \sum_{i=1}^n \alpha_i \mathbf{v}_i^T$ and $\alpha_i \neq 0$ for all $1 \leq i \leq n$.*

Proof. Note that, $\forall\ 1 \leq i \leq n$, $\mathcal{W}_i$ is a vector space of dimension $n-1$ and $\mathcal{V} \neq \mathcal{W}_i$; otherwise, $\mathbf{v}_j^T \in \mathcal{V}$ for all $j \neq i$, which contradicts the assumption of $\mathbf{v}_j^T \notin \mathcal{V}$ for all j. Since $\mathcal{V}$ and $\mathcal{W}_i$ are vector spaces of dimension $n-1$ for all i and $\cup_{i=1}^n \mathcal{W}_i$ is not a vector space, we have $\mathcal{V} \not\subset \cup_{i=1}^n \mathcal{W}_i$. This fact, together with $\{\mathbf{v}_1^T, \cdots, \mathbf{v}_n^T\}$ as a basis, implies that there exists a nonzero $\mathbf{v}^T \in \mathcal{V}$ such that $\mathbf{v}^T = \sum_{i=1}^n \alpha_i \mathbf{v}_i^T$ with $\alpha_i \neq 0$ for all i; otherwise, each $\mathbf{v}^T \in \mathcal{V}$ will belong $\mathcal{W}_i$ for some i, which contradicts $\mathcal{V} \not\subset \cup_{i=1}^n \mathcal{W}_i$. □

Lemma 2.1.4. *Let $\mathbf{c} \in \mathrm{I\!R}^{1 \times n}$ and $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}\}$ be LI. Additionally, $\mathbf{q}_n^T := \alpha_c \mathbf{c} + \sum_{j=1}^{n-1} \alpha_j \mathbf{q}_j^T$, $\alpha_c \neq 0$ and $\alpha_j \neq 0$ for all $j = 1, \cdots, n-1$. Then*

(i) $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_n^T\}$ *are LI.*

(ii) $\forall i \in \{1, \cdots, n\}$*, the* n *vectors* $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}\}$ *are LI.*

(iii) $\forall i \in \{1, \cdots, n\}$, $[\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp \not\subset (\mathbf{c})^\perp$.

Proof. (i) Suppose that $\sum_{j=1}^n k_j \mathbf{q}_j^T = \mathbf{0}^T$. Inserting the expression for $\mathbf{q}_n^T$ into the equation above yields $\sum_{j=1}^{n-1}(k_j + k_n\alpha_j)\mathbf{q}_j^T + k_n\alpha_c\mathbf{c} = \mathbf{0}^T$. Because $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}\}$ are LI, we have $k_n\alpha_c = 0$ and $k_j + k_n\alpha_j = 0$ for $1 \le j \le n-1$. Because $\alpha_c \neq 0$, we have $k_n = 0$ and $k_j = 0$ for $1 \le j \le n-1$. Thus, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_n^T\}$ are LI.

(ii) The case of $i = n$ is true by assumption. Suppose that $i < n$ and $\sum_{j \neq i} k_j \mathbf{q}_j^T + k_c\mathbf{c} = \mathbf{0}^T$. Inserting $\mathbf{q}_n^T$ into the equation yields $\sum_{j\neq i}^{n-1}(k_j + k_n\alpha_j)\mathbf{q}_j^T + k_n\alpha_i\mathbf{q}_i^T + (k_n\alpha_c + k_c)\mathbf{c} = \mathbf{0}^T$. Since $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}\}$ are LI and $\alpha_i \neq 0$, the coefficient of $\mathbf{q}_i^T$ yields $k_n = 0$ and thus $k_c = 0$ (from the coefficient of $\mathbf{c}$), and $k_j = 0$ for all $j \le n-1$ where $j \neq i$. Thus, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}\}$ are LI.

(iii) Suppose, on the contrary, that $[\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp \subset \mathbf{c}^\perp$. Then any nonzero vector $\mathbf{p} \in [\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp$ has the properties $\mathbf{c}\mathbf{p} = 0$ and $\mathbf{q}_j^T\mathbf{p} = 0$ for all $j \neq i$. Since, by (ii), $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}\}$ is a basis for $\mathbb{R}^{n*}$, it follows that $\mathbf{p}$ must be a zero vector, which contradicts the fact that $[\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp$ is a vector space of dimension 1. This finding proves that $[\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp \not\subset \mathbf{c}^\perp$. □

Till this end, we shall be able to derive the following main theorem in Chapter 2.

2.2 Theorem of Existence Conditions

Theorem 2.2.1.

(1) $\mathcal{A}^{si}_{\mathbf{xf}} \neq \emptyset$
$\Leftrightarrow$ "*$C\mathbf{x} \neq \mathbf{0}$ or $\mathbf{f} \neq \mathbf{0}$*"
$\Leftrightarrow$"*SDRE (1.19) admits a unique, symmetric, and stabilizing $P \geq 0$*".

(2) $\mathcal{A}^{sl}_{\mathbf{xf}} \neq \emptyset$
$\Leftrightarrow$ "*$C\mathbf{x} \neq \mathbf{0}$ or $\mathbf{f} \neq \mu\mathbf{x}$, for some $\mu \leq 0$*"
$\Leftrightarrow$ "*SDRE (1.19) admits a unique, symmetric, and stabilizing $P > 0$*".

(3) $\mathcal{A}^{so}_{\mathbf{xf}} \neq \emptyset$
$\Leftrightarrow$ "*$C\mathbf{x} \neq \mathbf{0}$ or $\{\mathbf{x}, \mathbf{f}\}$ are linearly independent (LI)*"
$\Rightarrow$"*SDRE (1.19) admits a unique, symmetric, and stabilizing $P \geq 0$*" *(resp., $\Rightarrow$"SDDRE (1.16) admits a unique and symmetric $P \geq 0$").*

Proof. The proof is divided into three parts as follows:

(i) Proof of $\mathcal{A}^{so}_{\mathbf{xf}} \neq \emptyset \Leftrightarrow$ "$C\mathbf{x} \neq \mathbf{0}$ or $\{\mathbf{x}, \mathbf{f}\}$ are LI" .

(ii) Proof of $\mathcal{A}^{sl}_{\mathbf{xf}} \neq \emptyset \Leftrightarrow$ "$C\mathbf{x} \neq \mathbf{0}$ or $\mathbf{f} \neq \mu\mathbf{x}$, for some $\mu \leq 0$" and similar extension to the case of $\mathcal{A}^{si}_{\mathbf{xf}} \neq \emptyset$.

(iii) The remaining results are adopted and summarized from [111, 113, 261].

PART (i): Divide the proof into the following four cases,

Case 1: ($\{\mathbf{x}, \mathbf{f}\}$ are LI and $C[\mathbf{x}, \mathbf{f}] \neq \mathbf{0}$)
Because $\{\mathbf{x}, \mathbf{f}\}$ are LI, we may choose $\lambda_1, \cdots, \lambda_n \in \mathbb{R}^-$ such that any two of the n vectors $\{\lambda_i\mathbf{x} - \mathbf{f} \mid i = 1, \cdots, n\}$ are not collinear. Additionally, $C[\mathbf{x}, \mathbf{f}] \neq \mathbf{0} \Rightarrow$ $\exists$ a nonzero row vector $\mathbf{c}$ of C with $\mathbf{c} \notin [\mathbf{x}, \mathbf{f}]^\perp$. It follows that $\mathbf{c}(\lambda_i\mathbf{x} - \mathbf{f}) \neq 0$ for all $i = 1, \cdots, n$. If $n > 2$, we may easily choose $\mathbf{q}_i^T \in (\lambda_i\mathbf{x} - \mathbf{f})^\perp$, $1 \leq i \leq n-2$, satisfying $\mathbf{q}_1^T \notin (\lambda_{n-1}\mathbf{x} - \mathbf{f})^\perp$, $\mathbf{q}_1^T(\lambda_n\mathbf{x} - \mathbf{f}) > 0$, $\mathbf{q}_i^T(\lambda_n\mathbf{x} - \mathbf{f}) \geq 0$ for $i = 2, \cdots, n-2$ and $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-2}^T, \mathbf{c}\}$ are LI because $\dim((\lambda_i\mathbf{x} - \mathbf{f})^\perp) = n-1$ for all i. Since $\mathbf{q}_1^T \notin (\lambda_{n-1}\mathbf{x} - \mathbf{f})^\perp$ and $\dim((\lambda_{n-1}\mathbf{x} - \mathbf{f})^\perp) = n-1$, it follows from Lemma 2.1.2 that there exists a $\mathbf{q}_{n-1}^T \in (\lambda_{n-1}\mathbf{x} - \mathbf{f})^\perp$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}\}$ are LI and $\mathbf{q}_{n-1}^T(\lambda_n\mathbf{x} - \mathbf{f}) \geq 0$. Define $\mathbf{q}_n^T = \alpha\mathbf{c} + \sum_{i=1}^{n-1}\mathbf{q}_i^T$, $\alpha = -[\sum_{i=1}^{n-1}\mathbf{q}_i^T(\lambda_n\mathbf{x} - \mathbf{f})]/[\mathbf{c}(\lambda_n\mathbf{x} - \mathbf{f})]$. Clearly, $\alpha \neq 0$ because $\mathbf{c}(\lambda_n\mathbf{x} - \mathbf{f}) \neq 0$, $\mathbf{q}_1^T(\lambda_n\mathbf{x} - \mathbf{f}) > 0$ and $\mathbf{q}_i^T(\lambda_n\mathbf{x} - \mathbf{f}) \geq 0$ for $2 \leq i \leq n-1$. Moreover, it is found that $\mathbf{q}_n^T(\lambda_n\mathbf{x} - \mathbf{f}) = 0$, therefore, from (i) of Lemma 2.1.1, $A\mathbf{x} = \mathbf{f}$. To prove observability, it is noted that $M^{-1}M = I$, where M is given by Eq. (2.2),

yields $\mathbf{p}_i \in [\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp$ for all i. It follows from Lemma 2.1.4 that $\mathbf{p}_i \notin \mathbf{c}^\perp$ for all i. This fact combined with $C^\perp \subset \mathbf{c}^\perp$ and (ii) of Lemma 2.1.1 implies that (A, C) is observable. Finally, (A, B) is stabilizable because $\lambda_i \in \mathbb{R}^-$ for all i. Thus, $A \in \mathcal{A}_{\mathbf{xf}}^{so} \neq \emptyset$. If $n = 2$, the proof can be similarly derived if we choose $\mathbf{q}_1^T \in (\lambda_1 \mathbf{x} - \mathbf{f})^\perp$, $\mathbf{q}_1^T(\lambda_2 \mathbf{x} - \mathbf{f}) > 0$, $\mathbf{q}_2^T = \alpha \mathbf{c} + \mathbf{q}_1^T$ and $\alpha = -\frac{\mathbf{q}_1^T(\lambda_2 \mathbf{x} - \mathbf{f})}{\mathbf{c}(\lambda_2 \mathbf{x} - \mathbf{f})}$.

Case 2: ($\{\mathbf{x}, \mathbf{f}\}$ are LI but $C[\mathbf{x}, \mathbf{f}] = \mathbf{0}$)
Let $\mathbf{c}$ be a nonzero row vector of C. Thus, $\mathbf{c} \in [\mathbf{x}, \mathbf{f}]^\perp$ because $C[\mathbf{x}, \mathbf{f}] = \mathbf{0}$. Additionally, because $\{\mathbf{x}, \mathbf{f}\}$ are LI we may choose, as in Case 1, $\lambda_1, \cdots, \lambda_n \in \mathbb{R}^-$ such that any two of the n vectors $\{\lambda_i \mathbf{x} - \mathbf{f} \mid i = 1, \cdots, n\}$ are not collinear. Suppose that $n > 2$. Choose $\mathbf{q}_1 \in \text{span}\{\mathbf{x}, \mathbf{f}\}$, $\mathbf{q}_1 \neq \mathbf{0}$ and $\mathbf{q}_1^T(\lambda_1 \mathbf{x} - \mathbf{f}) = 0$. Thus, $\mathbf{q}_1^T(\lambda_i \mathbf{x} - \mathbf{f}) \neq 0$ for all $2 \leq i \leq n$, i.e., $\mathbf{q}_1^T \notin [\lambda_i \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$ for all $2 \leq i \leq n$. Since both $[\lambda_2 \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$ and $[\lambda_n \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$ are vector spaces of dimension $n-2$ and $[\lambda_2 \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp \not\subset [\lambda_n \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$, it follows from Lemma 2.1.2 that there exists $\mathbf{q}_2^T \in [\lambda_2 \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp \backslash [\lambda_n \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$ such that $\{\mathbf{q}_1^T, \mathbf{q}_2^T\}$ are LI. Continuing this process, using $\mathbf{q}_1^T \notin [\lambda_i \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$ and Lemma 2.1.2, we can find that $\mathbf{q}_i^T \in [\lambda_i \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp \backslash [\lambda_n \mathbf{x} - \mathbf{f}, \mathbf{c}^T]^\perp$, $i = 3, \cdots, n-1$, such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T\}$ are LI. Moreover, it is easy to find a vector $\mathbf{w}^T \notin (\lambda_n \mathbf{x} - \mathbf{f})^\perp$ such that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{w}^T\}$ are LI. Because $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{w}^T\}$ is a basis of $\mathbb{R}^n$ and $\mathbf{w}^T$, it follows that $\mathbf{q}_i^T \notin (\lambda_n \mathbf{x} - \mathbf{f})^\perp$ for all $1 \leq i \leq n-1$, and by Lemma 2.1.3 there exists a $\mathbf{v}^T \in (\lambda_n \mathbf{x} - \mathbf{f})^\perp$ such that $\mathbf{v}^T = \sum_{i=1}^{n-1} \alpha_i \mathbf{q}_i^T + \alpha_n \mathbf{w}$ and $\alpha_i \neq 0$ for all i. Since both $\mathbf{c}$ and $\mathbf{v}^T$ belong to $(\lambda_n \mathbf{x} - \mathbf{f})^\perp$, we have $\mathbf{q}_n^T := \mathbf{c} + \mathbf{v}^T \in (\lambda_n \mathbf{x} - \mathbf{f})^\perp$. Thus, $\mathbf{q}_i^T \in (\lambda_i \mathbf{x} - \mathbf{f})^\perp$ for all i and, by (i) of Lemma 2.1.1, $A\mathbf{x} = \mathbf{f}$. Additionally, from the selection of $\mathbf{q}_n^T$, the fact that $\mathbf{p}_i \in [\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T]^\perp$ and Lemma 2.1.4, we have $\mathbf{p}_i \notin \mathbf{c}^\perp$ for all i. Combined with $C^\perp \subset \mathbf{c}^\perp$ and (ii) of Lemma 2.1.1 yields that (A, C) is observable. Finally, (A, B) is stabilizable because $\lambda_i \in \mathbb{R}^-$ for all i. Thus, $A \in \mathcal{A}_{\mathbf{xf}}^{so} \neq \emptyset$. The case for $n = 2$ can be similarly derived if we choose $\mathbf{q}_1^T \in (\lambda_1 \mathbf{x} - \mathbf{f})^\perp \backslash (\lambda_2 \mathbf{x} - \mathbf{f})^\perp$.

Case 3: ($\{\mathbf{x}, \mathbf{f}\}$ are linearly dependent (LD) but $C\mathbf{x} \neq 0$)
Let $\mathbf{c}$ be a nonzero row vector of C such that $\mathbf{cx} \neq 0$, and $\mathbf{b}$ be a nonzero column vector of B. We choose $n-1$ distinct real numbers $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}^-$, and $n-1$ LI row vectors $\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T \in \mathbf{x}^\perp$. These imply that $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T, \mathbf{c}\}$ are LI because $\mathbf{c} \notin \mathbf{x}^\perp$. If $\{\mathbf{x}, \mathbf{b}\}$ are LD (i.e., $\mathbf{x}^\perp = \mathbf{b}^\perp$), we choose $\mathbf{q}_n^T := \alpha \mathbf{c} + \sum_{i=1}^{n-1} \mathbf{q}_i^T$, where $\alpha \neq 0$. It follows that $\mathbf{q}_n^T \notin \mathbf{b}^\perp$, and thus $\mathbf{q}_n^T \notin B^\perp$, because $B^\perp \subset \mathbf{b}^\perp$. On the other hand, if $\{\mathbf{x}, \mathbf{b}\}$ are LI, the above-mentioned $\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T$ may be chosen from $\mathbf{x}^\perp$ satisfying $\mathbf{q}_1^T \mathbf{b} > 0$ and $\mathbf{q}_i^T \mathbf{b} \geq 0$ for all $i = 2, \cdots, n-1$. It follows that $\sum_{i=1}^{n-1} \mathbf{q}_i^T \mathbf{b} > 0$, and therefore there exists a nonzero constant α such that $(\alpha \mathbf{c} + \sum_{i=1}^{n-1} \mathbf{q}_i^T)\mathbf{b} = \alpha \mathbf{cb} + \sum_{i=1}^{n-1} \mathbf{q}_i^T \mathbf{b} \neq 0$ regardless of if $\mathbf{cb}$ is zero. Furthermore, we also choose $\mathbf{q}_n^T := \alpha \mathbf{c} + \sum_{i=1}^{n-1} \mathbf{q}_i^T$

as before. Clearly, $\mathbf{q}_n^T \notin B^\perp$ because $\mathbf{q}_n^T \notin \mathbf{b}^\perp$ and $B^\perp \subset \mathbf{b}^\perp$. Finally, we choose λ_n such that $\mathbf{q}_n^T(\lambda_n \mathbf{x} - \mathbf{f}) = 0$. From these discussions, we have $\mathbf{q}_i^T(\lambda_i \mathbf{x} - \mathbf{f}) = 0$ for all $i = 1, \cdots, n$, which implies from (i) of Lemma 2.1.1 that $A\mathbf{x} = \mathbf{f}$. Additionally, due to the selection of $\mathbf{q}_n^T$ and (ii) of Lemma 2.1.4, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T, \mathbf{c}\}$ are LI for any $i = 1, \cdots, n$. Combined with (iii) of Lemma 2.1.4 and the fact that $\mathbf{p}_i \in \{\mathbf{q}_1^T, \cdots, \mathbf{q}_{i-1}^T, \mathbf{q}_{i+1}^T, \cdots, \mathbf{q}_n^T\}^\perp$ leads to $\mathbf{c}\mathbf{p}_i \neq 0$ for all i, and thus $C\mathbf{p}_i \neq 0$ for all i. Hence, by (ii) of Lemma 2.1.1, (A, C) is observable. Finally, because $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}^-$ and $\mathbf{q}_n^T \notin B^\perp$, (A, B) is stabilizable by (iii) of Lemma 2.1.1. Thus, $A \in \mathcal{A}_{\mathbf{xf}}^{so} \neq \emptyset$.

Case 4: ($\{\mathbf{x}, \mathbf{f}\}$ are LD and $C\mathbf{x} = 0$)
Because $\{\mathbf{x}, \mathbf{f}\}$ are LD, we have $\mathbf{f} = \lambda \mathbf{x}$ for some $\lambda \in \mathbb{R}$. Suppose that there exists A such that $A\mathbf{x} = \mathbf{f}$. Then $A\mathbf{x} = \mathbf{f} = \lambda \mathbf{x}$, i.e., $(\lambda, \mathbf{x})$ is an eigenpair of A. This fact combined with the condition $C\mathbf{x} = 0$ yields that (A, C) is unobservable (by the PBH test [43]) and $\mathcal{A}_{\mathbf{xf}}^{so} = \emptyset$.

PART (ii): Since $\mathcal{A}^l = (-\mathcal{A})^d := \{-A \mid A \in \mathcal{A}^d\}$ and for brevity of notation, we will show the counterpart $\mathcal{A}_{\mathbf{xf}}^{sd} \neq \emptyset \Leftrightarrow$ "$C\mathbf{x} \neq \mathbf{0}$ or $\mathbf{f} \neq \mu \mathbf{x}$, for some $\mu \geq 0$", and the cases of $\mathcal{A}_{\mathbf{xf}}^{sl}$ and $\mathcal{A}_{\mathbf{xf}}^{si}$ are similarly obtained. It was shown in **PART (i)** that $\mathcal{A}_{\mathbf{xf}}^{so} \neq \emptyset$ for the first three cases. It follows that $\mathcal{A}_{\mathbf{xf}}^{sd} \neq \emptyset$ for the first three cases because observability implies detectability. Thus, it remains to consider the fourth case, where $\{\mathbf{x}, \mathbf{f}\}$ are LD and $C\mathbf{x} = 0$. If $\mathbf{f} = \lambda \mathbf{x}$ with $\lambda \geq 0$, then $(\lambda, \mathbf{x})$ is an unstable eigenpair of A because $A\mathbf{x} = \mathbf{f} = \lambda \mathbf{x}$. Therefore, by the PBH test [43], (A, C) is undetectable and $\mathcal{A}_{\mathbf{xf}}^{sd} = \emptyset$. On the contrary, suppose that $\mathbf{f} = \lambda \mathbf{x}$ and $\lambda < 0$. Choose $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}^-$, $\lambda_n = \lambda$, $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T\}$ to be $n-1$ LI vectors of $\{\mathbf{x}\}^\perp$ and $\mathbf{q}_n^T = \mathbf{x}^T$. We have, from (i) of Lemma 2.1.1, that $A \in \mathcal{A}_{\mathbf{xf}}^{sd}$ because all of the eigenvalues of A are selected to have negative real part. $\square$

2.3 Easy Construction of A Feasible SDC Matrix (Solution to Problem 2)

Based on the derivation in Section 2.2, a real and diagonalizable SDC matrix $A \in \mathcal{A}^{s\alpha}_{\mathbf{xf}}$, $\alpha = o, d, l, i$, can be easily constructed, if the associated existence conditions are satisfied. In the following, the construction algorithm is described and, the case of $\mathcal{A}^{so}_{\mathbf{xf}}$ is omitted since it can be readily found in part (i) of the proof of Theorem 2.2.1.

Algorithm 2.3.1: Select an SDC matrix in $\mathcal{A}^{sd}_{\mathbf{xf}}$ (resp. $\mathcal{A}^{s\alpha}_{\mathbf{xf}}$, $\alpha = i, l$)

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input : $\mathbf{x}, \mathbf{f}, B$ and C
output : $A \in \mathcal{A}^{sd}_{\mathbf{xf}}$ (resp. $A \in \mathcal{A}^{s\alpha}_{\mathbf{xf}}$, $\alpha = i, l$)

Choose $\lambda_1, \cdots, \lambda_{n-1} \in \mathbb{R}^-$.
if $\{\mathbf{x}, \mathbf{f}\}$ are linearly independent (LI) **then**
    Choose $\lambda_n \in \mathbb{R}^-$ with $\lambda_n \neq \lambda_{n-1}$;
    Choose $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-2}^T\}$ being a LI subset of $\{\mathbf{x}, \mathbf{f}\}^\perp$;
    Choose $\{\mathbf{q}_{n-1}^T, \mathbf{q}_n^T\} \subseteq \text{span}\{\mathbf{x}, \mathbf{f}\}$ satisfying $\mathbf{q}_i^T(\lambda_i \mathbf{x} - \mathbf{f}) = 0$ for $i = n-1, n$;
else
    Choose $\lambda_n = \lambda$ (resp. $\lambda_n = -\lambda$), where $\mathbf{f} = \lambda \mathbf{x}$;
    Choose $\{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T\}$ being a LI subset of $\{\mathbf{x}\}^\perp$;
    Choose $\mathbf{q}_n^T \notin B^\perp \cup \{\mathbf{x}\}^\perp$;
$A = M^{-1}DM$, where $M = M(\mathbf{x}) = [\mathbf{q}_1, \cdots, \mathbf{q}_n]^T$ and $D = D(\mathbf{x}) = diag[\lambda_1, \cdots, \lambda_n]$.

Proof. The proof is divided into two parts: (i) $\mathcal{A}^{sd}_{\mathbf{xf}}$; (ii) $\mathcal{A}^{s\alpha}_{\mathbf{xf}} \neq \emptyset$, $\alpha = i, l$.

PART (i):

(1) It is clear from the choices that $\mathbf{q}_i^T(\lambda_i \mathbf{x} - \mathbf{f}) = 0$ for all i. Thus, by (i) of Lemma 2.1.1, $A\mathbf{x} = \mathbf{f}$. Moreover, since $\lambda_i \in \mathbb{R}^-$ for all i, (A, B) is stabilizable and (A, C) is detectable.

(2) Since $\{\mathbf{x}, \mathbf{f}\}$ are LD, we have $\lambda_i \mathbf{x} - \mathbf{f} = k_i \mathbf{x}$ for some $k_i \in \mathbb{R}$, $i = 1, \cdots, n-1$, and $\lambda_n \mathbf{x} - \mathbf{f} = \mathbf{0}$ (since $\lambda_n = \lambda$). It follows from the choices of $\mathbf{q}_i^T$ that $\mathbf{q}_i^T(\lambda_i \mathbf{x} - \mathbf{f}) = 0$ for all i. Thus, by (i) of Lemma 2.1.1, $A\mathbf{x} = \mathbf{f}$. It remains to verify the stabilizability and detectability conditions for the following two cases of "$\lambda_n < 0$" and "$\lambda_n \geq 0$, but $C\mathbf{x} \neq \mathbf{0}$." First, if $\lambda_n < 0$, then all the eigenvalues of A are negative and the stabilizability and detectability conditions hold. Next, suppose that $\lambda_n \geq 0$. Then, by the choice of $\mathbf{q}_n^T \notin B^\perp$ and (iii) of Lemma 2.1.1,

we have (A, B) is stabilizable. Moreover, due to $M^{-1}M = I$, where M is given by (2.2), and the choice of $\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T$, we have $\mathbf{p}_n \in \{\mathbf{q}_1^T, \cdots, \mathbf{q}_{n-1}^T\}^\perp = \text{span}\{\mathbf{x}\}$. Since, in this case, $C\mathbf{x} \neq \mathbf{0}$, we have $C\mathbf{p}_n \neq \mathbf{0}$. Thus, by (ii) of Lemma 2.1.1, (A, C) is detectable and the result is obtained.

PART (ii): Since $\mathcal{A}^l = (-\mathcal{A})^d := \{-A \mid A \in \mathcal{A}^d\}$, the case for $\mathcal{A}_{\mathbf{xf}}^{sl}$ can be similarly derived from **PART (i)** by treating the pair $(A, \mathbf{x}, \mathbf{f})$ as $(-A, \mathbf{x}, -\mathbf{f})$. On the other hand, since $\mathcal{A}^l \subseteq \mathcal{A}^i$, together with the fact that $\mathcal{A}^l = \emptyset \Leftrightarrow$ "$C\mathbf{x} = \mathbf{0}$ and $\mathbf{f} = \mu\mathbf{x},\ \mu \leq 0$", we only need to consider the following two cases for $\mathcal{A}_{\mathbf{xf}}^{si}$. If $\mu = 0$, then by (2) of Theorem 2.2.1, $\mathcal{A}^i = \mathcal{A}_{\mathbf{xf}}^{si} = \emptyset$; If $\mu < 0$, then by (2) of Theorem 2.2.1, $\mathcal{A}^i \neq \emptyset$, and by similar arguments as **PART (i)**, the result thus follows. □

2.4 Concluding Remarks

In this chapter, necessary and sufficient conditions for the existence of possible SDC matrices are specified, such that the associated SDRE (1.19) (resp. SDDRE (1.16)) is solvable. These existence conditions are easy to verify, and a set of feasible SDC matrices are also presented explicitly when the existence conditions hold. These analytic results may provide a means to successfully continue the SDRE/SDDRE scheme at those states where the associated SDRE/SDDRE (1.19/1.16) is unsolvable, yet the presented existence conditions hold there. As a step further, how to specifically represent all feasible SDC matrices will be addressed in the next chapter.

Chapter 3

Parameterizations of All Feasible SDC Matrices [1]

Once the existence conditions stated in Theorem 2.2.1 are satisfied, this chapter will present a parametrization of their solution matrices. Without loss of any generality, we assume in this chapter that B (resp. C) has full column (resp. row) rank. Moreover, define $A_p = \frac{1}{||\mathbf{x}||^2}\mathbf{f}\mathbf{x}^T$. It is clear that $A_p\mathbf{x} = \mathbf{f}$ and

$$\mathcal{A}_{\mathbf{xf}} = \left\{ A_p + K\mathbf{x}_\perp \mid K \in \mathbb{R}^{n\times(n-1)} \right\}. \tag{3.1}$$

Obviously, $\mathcal{A}_{\mathbf{xf}} \subset \mathbb{R}^{n\times n}$ is a linear variety (i.e., a subspace through a translation) of dimension $n^2 - n$ and K describes the $n^2 - n$ free parameters. Besides, A_p has minimum Frobenius norm among the matrices in $\mathcal{A}_{\mathbf{xf}}$, as explained in Appendix A.1.

3.1 Preliminary Results (Solution to Problem 1)

We present the following two results which can reduce the dimension of checking system controllability, stabilizability, observability and detectability.

Lemma 3.1.1. *Let* $\bar{A} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{pmatrix}$ *and* $\bar{B} = \begin{pmatrix} 0 \\ \bar{B}_2 \end{pmatrix}$, *where* $\bar{B}_2 \in \mathbb{R}^{p\times p}$ *is a nonsingular matrix,* $\bar{A}_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ *and* $\bar{A}_{22} \in \mathbb{R}^{p\times p}$. *Then* $(\bar{A}, \bar{B})$ *is*

[1] Journal version at [153, 159]

controllable (resp., stabilizable) $\Leftrightarrow (\bar{A}_{11}, \bar{A}_{12})$ *is controllable (resp., stabilizable). In particular, when* $p < n$ *and* $\bar{A}_{12} = 0$*, then* $(\bar{A}, \bar{B})$ *is uncontrollable, and it is stabilizable* $\Leftrightarrow \lambda(\bar{A}_{11}) \subset \mathrm{C}^-$*.*

Proof. Consider only the case of controllability, while the stabilizability result can be similarly derived. By PBH test [43], $(\bar{A}, \bar{B})$ is controllable

$\Leftrightarrow \forall \lambda \in \mathrm{C}$, $\mathrm{rank}\begin{pmatrix} \bar{A}_{11} - \lambda I_{n-p} & \bar{A}_{12} & 0 \\ \bar{A}_{21} & \bar{A}_{22} - \lambda I_p & \bar{B}_2 \end{pmatrix} = n.$

$\Leftrightarrow \forall \lambda \in \mathrm{C}$, $\mathrm{rank}\begin{pmatrix} \bar{A}_{11} - \lambda I_{n-p} \ \vdots \ \bar{A}_{12} \end{pmatrix} = n - p$, because $\bar{B}_2$ is a nonsingular matrix.

$\Leftrightarrow (\bar{A}_{11}, \bar{A}_{12})$ is a controllable pair.

□

It is known that controllability (resp., stabilizability) and observability (resp., detectability) are dual concepts, that is, (A, C) is observable (resp., detectable) iff (A^T, C^T) is controllable (resp., stabilizable). Therefore, Lemma 3.1.1 can be easily extended to the cases of observability and detectability as stated below:

Corollary 3.1.1. *Let* $\bar{A}$ *be partitioned in the form given by Lemma 3.1.1 with* $\bar{A}_{11} \in \mathbb{R}^{(n-q)\times(n-q)}$ *and* $\bar{A}_{22} \in \mathbb{R}^{q\times q}$*.* $\bar{C} = [0, \bar{C}_2]$*, where* $\bar{C}_2 \in \mathbb{R}^{q\times q}$ *is a nonsingular matrix. Then*

(i) $(\bar{A}, \bar{C})$ *is observable (resp., detectable)* $\Leftrightarrow (\bar{A}_{11}, \bar{A}_{21})$ *is observable (resp., detectable).*

(ii) $(\bar{A}, \bar{C})$ *has no unobservable mode on the* $j\omega$*-axis (resp., LHP and* $j\omega$*-axis)* $\Leftrightarrow (\bar{A}_{11}, \bar{A}_{21})$ *has no unobservable mode on the* $j\omega$*-axis (resp., LHP and* $j\omega$*-axis).*

In particular, when $q < n$ *and* $\bar{A}_{21} = 0$*, then* $(\bar{A}, \bar{C})$ *is unobservable and*

(iii) $(\bar{A}, \bar{C})$ *is detectable* $\Leftrightarrow \lambda(\bar{A}_{11}) \subset \mathrm{C}^-$*.*

(iv) $(\bar{A}, \bar{C})$ *has no unobservable mode on the* $j\omega$*-axis (resp., LHP and* $j\omega$*-axis)* $\Leftrightarrow \bar{A}_{11}$ *has no eigenvalue on the* $j\omega$*-axis (resp., LHP and* $j\omega$*-axis).*

From Lemma 3.1.1 and Corollary 3.1.1, it is easy to verify that the sets $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$ and $\mathcal{A}^d$ are both open and dense in $\mathbb{R}^{n\times n}$, which agree with the results given by [74] and [157]. To apply Lemma 3.1.1 and Corollary 3.1.1, we have to transform (A, B) (resp., (A, C)) into the form of $(\bar{A}, \bar{B})$ (resp., $(\bar{A}, \bar{C})$) as stated in Lemma 3.1.1 (resp., Corollary 3.1.1). Such coordinate transformation can be chosen to be orthogonal as the form of (3.2) below:

$$\mathbf{x} = M_B \bar{\mathbf{x}} \quad \text{(resp., } \mathbf{x} = M_C \bar{\mathbf{x}}\text{)} \tag{3.2}$$

where M_B and M_C are orthogonal matrices. A candidate of M_B (resp., M_C) can be determined by the QR factorization scheme for B (resp., C^T) and then interchanges the position of the first p (resp., q) columns with the last $n-p$ (resp., $n-q$) columns.

To explore the relation between $\mathcal{A}_{\mathbf{xf}}$ in the original coordinate and $\mathcal{A}_{\bar{\mathbf{x}}\bar{\mathbf{f}}}$ in the transformed coordinate using the coordinate transformation $M = M_B$ or $M = M_C$ given by Eq. (3.2), it is noted that $\bar{A} = M^T A M$ and $\bar{\mathbf{x}} = M^T \mathbf{x}$, because M is an orthogonal matrix. By letting $\bar{\mathbf{f}} = M^T \mathbf{f}$, we have $A\mathbf{x} = \mathbf{f} \Leftrightarrow \bar{A}\bar{\mathbf{x}} = \bar{\mathbf{f}}$. Besides, because $\mathbf{x}_\perp$ and $\bar{\mathbf{x}}_\perp$ denote two selected $(n-1) \times n$ state-dependent matrices satisfying $\mathbf{x}_\perp \mathbf{x} = \mathbf{0}$ and $\bar{\mathbf{x}}_\perp \bar{\mathbf{x}} = \mathbf{0}$, we choose $\bar{\mathbf{x}}_\perp = \mathbf{x}_\perp M$. It follows that $\bar{A} = \frac{1}{\bar{\mathbf{x}}^T\bar{\mathbf{x}}}\bar{\mathbf{f}}\bar{\mathbf{x}}^T + \bar{K}\bar{\mathbf{x}}_\perp = M^T \left[\frac{1}{\mathbf{x}^T\mathbf{x}}\mathbf{f}\mathbf{x}^T + K\mathbf{x}_\perp\right] M = M^T A M$ if we choose $K, \bar{K} \in \mathbb{R}^{n\times(n-1)}$ satisfying $K = M\bar{K}$. Moreover, since controllability, observability, stabilizability and detectability are invariant under equivalence transformation [43], we have the following result:

Theorem 3.1.1. *Let M_B (resp., M_C) be an orthogonal matrix given by Eq. (3.2) such that $B = M_B\bar{B}$ (resp., $C^T = M_C\bar{C}^T$) and $\bar{B}$ (resp., $\bar{C}$) is given by Lemma 3.1.1 (resp., Corollary 3.1.1), $\bar{\mathbf{x}}_\perp = \mathbf{x}_\perp M_B$ and $K = M_B\bar{K}$ (resp., $\bar{\mathbf{x}}_\perp = \mathbf{x}_\perp M_C$ and $K = M_C\bar{K}$). Besides, $A = \frac{1}{||\mathbf{x}||^2}\mathbf{f}\mathbf{x}^T + K\mathbf{x}_\perp \in \mathcal{A}_{\mathbf{xf}}$ and $\bar{A} = \frac{1}{||\bar{\mathbf{x}}||^2}\bar{\mathbf{f}}\bar{\mathbf{x}}^T + \bar{K}\bar{\mathbf{x}}_\perp \in \mathcal{A}_{\bar{\mathbf{x}}\bar{\mathbf{f}}}$. Then*

(i) *(A, B) is controllable (resp., (A, C) is observable) $\Leftrightarrow$ $(\bar{A}, \bar{B})$ is controllable (resp., $(\bar{A}, \bar{C})$ is observable).*

(ii) *(A, B) is stabilizable (resp., (A, C) is detectable) $\Leftrightarrow$ $(\bar{A}, \bar{B})$ is stabilizable (resp., $(\bar{A}, \bar{C})$ is detectable).*

(iii) *(A, C) has has no unobservable mode on the $j\omega$-axis (resp., LHP and $j\omega$-axis) $\Leftrightarrow$ $(\bar{A}, \bar{C})$ has no unobservable mode on the $j\omega$-axis (resp., LHP and $j\omega$-axis).*

After deriving the sets $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^c$, $\mathcal{A}^s$, $\mathcal{A}^o$ and $\mathcal{A}^d$, it is clear that $\mathcal{A}^{so}_{\mathbf{xf}} := \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^s \cap \mathcal{A}^o$ and $\mathcal{A}^{sd}_{\mathbf{xf}} := \mathcal{A}_{\mathbf{xf}} \cap \mathcal{A}^s \cap \mathcal{A}^d$, respectively. Moreover, Problem 1 can be easily solved via the combination of results in this section, which alleviates the computational burden to determine the property of the solution of the corresponding SDRE (resp., SDDRE) by direct calculation.

3.2 Main result of Parameterizations (Solution to Problem 3)

At first, we consider the case of rank(B), rank(C) $\geq n-1$. When rank(B) $= n$ (resp., rank(C) $= n$), (A, B) (resp., (A, C)) is controllable (resp., observable). Thus, $\mathcal{A}^c_{\mathbf{xf}} = \mathcal{A}^s_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$ (resp., $\mathcal{A}^o_{\mathbf{xf}} = \mathcal{A}^d_{\mathbf{xf}} = \mathcal{A}^l_{\mathbf{xf}} = \mathcal{A}^i_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$), which is a (n^2-n)-dimension linear variety.

Next, if rank(B) $= n-1$ (resp., rank(C) $= n-1$), the orthogonal matrix M_B (resp., M_C), given by Eq. (3.2), is selected to be $M_B = (B^T_\perp \vdots \tilde{B})$ (resp., $M_C = (C_\perp \vdots \tilde{C}^T)$), where the columns of $\tilde{B}$ (resp., $\tilde{C}^T$) form an orthonormal basis of the range space of B (resp., C^T). In this case, $\bar{A}^T_{12}, \bar{A}_{21} \in \mathrm{I\!R}^{\mathrm{n}-1}$ and $\bar{A}_{11} \in \mathrm{I\!R}$. To describe the structure of $\mathcal{A}^c_{\mathbf{xf}}$, $\mathcal{A}^s_{\mathbf{xf}}$, $\mathcal{A}^o_{\mathbf{xf}}$, $\mathcal{A}^d_{\mathbf{xf}}$, $\mathcal{A}^l_{\mathbf{xf}}$, and $\mathcal{A}^i_{\mathbf{xf}}$, we need the following Lemmas 3.2.1-3.2.3. Note that, to avoid any possible abuse of notation, for any $\mathbf{v} \in \mathrm{I\!R}^{\mathrm{n}}$ (resp. $\mathrm{I\!R}^{1\times\mathrm{n}}$), we denote $\mathbf{v}_\perp \in \mathrm{I\!R}^{(\mathrm{n}-1)\times\mathrm{n}}$ (resp. $\mathrm{I\!R}^{\mathrm{n}\times(\mathrm{n}-1)}$) as a matrix with orthonormal rows (resp. columns) such that $\mathbf{v}_\perp\mathbf{v} = 0$ (resp. $\mathbf{v}\mathbf{v}_\perp = 0$).

Lemma 3.2.1. *Let $\mathbf{v} \in \mathrm{I\!R}^{\mathrm{n}}\backslash\{\mathbf{0}\}$ and $M \in \mathrm{I\!R}^{\mathrm{n}\times(\mathrm{n}-1)}$ having full column rank. Then rank$(\mathbf{v}_\perp M) \geq n-2$. Moreover, $\mathbf{v}_\perp M$ is singular if and only if $\mathbf{v} \in$ span$\{M\}$.*

Proof. Given rank$(M) = n-1$, we have that
$\mathbf{v}_\perp M$ is singular
$\Leftrightarrow \exists\ \mathbf{y} \in \mathrm{I\!R}^{\mathrm{n}-1}\backslash\{\mathbf{0}\}$ such that $\mathbf{v}_\perp \mathrm{M}\mathbf{y} = \mathbf{0}$
$\Leftrightarrow\ \exists\ \mathbf{y} \in \mathrm{I\!R}^{\mathrm{n}-1}\backslash\{\mathbf{0}\}$ such that$\{\mathrm{M}\mathbf{y}, \mathbf{v}\}$ are LD
$\Leftrightarrow \mathbf{v} \in$ span$\{M\}$.
Hence rank$(M\mathbf{y}) = 1$ and rank$(\mathbf{v}_\perp M) = n-2$, $\forall \mathbf{y} \in \mathrm{I\!R}^{\mathrm{n}-1}\backslash\{\mathbf{0}\}$. On the contrary, if $\mathbf{v} \notin$ span$\{M\}$, then $\mathbf{v}_\perp M$ is nonsingular $\Leftrightarrow$ rank$(\mathbf{v}_\perp M) = n-1$. $\square$

Lemma 3.2.2. *Let $B \in \mathrm{I\!R}^{\mathrm{n}\times(\mathrm{n}-1)}$ having full column rank, $\boldsymbol{\xi} \in \mathrm{I\!R}^{\mathrm{n}-1}$, and $M \in \mathrm{I\!R}^{(\mathrm{n}-1)\times(\mathrm{n}-1)}\backslash\{0\}$. Define $\mathbf{m}_p \in \mathrm{I\!R}^{\mathrm{n}-1}$ to be such that $M^T\mathbf{m}_p = \boldsymbol{\xi}$. Then the set of $K \in \mathrm{I\!R}^{\mathrm{n}\times(\mathrm{n}-1)}$ such that $(B_\perp K)M = \boldsymbol{\xi}^T$ can be parameterized as $K = \left\{B^T_\perp(\mathbf{m}^T_p + \kappa\boldsymbol{\eta}^T) + \tilde{B}K' \mid \kappa \in \mathrm{I\!R}, \boldsymbol{\eta} \in \mathcal{N}(\mathrm{M}^\mathrm{T}) \text{ and } \mathrm{K}' \in \mathrm{I\!R}^{(\mathrm{n}-1)\times(\mathrm{n}-1)}\right\}$. In case M being nonsingular, we have $\mathbf{m}_p = \boldsymbol{\xi}M^{-T}$ and $\boldsymbol{\eta} = \mathbf{0}$.*

Proof. $(B_\perp K)M = \boldsymbol{\xi}^T \Leftrightarrow M^T(K^TB^T_\perp) = \boldsymbol{\xi} \Leftrightarrow K^TB^T_\perp = \{m_p + \kappa\boldsymbol{\eta} \mid \kappa \in \mathrm{I\!R},\ \boldsymbol{\eta} \in \mathcal{N}(M^T)\}$. Then in view of Eq. (1.21), we may similarly parameterize K^T and the result thus follows. $\square$

Lemma 3.2.3. *Let* $C \in \mathbb{R}^{(\mathrm{n}-1)\times \mathrm{n}}$ *having full row rank and* $\boldsymbol{\nu}, \boldsymbol{\xi} \in \mathbb{R}^{\mathrm{n}-1}$. *Define* $\mathbf{m}_p \in \mathbb{R}^{\mathrm{n}}$ *to be such that* $C\mathbf{m}_p = \boldsymbol{\xi}$. *Then the set of* $K \in \mathbb{R}^{\mathrm{n}\times(\mathrm{n}-1)}$ *such that* $CK\boldsymbol{\nu} = \boldsymbol{\xi}$ *can be parameterized as* $K = \left\{\frac{1}{||\boldsymbol{\nu}||^2}(\mathbf{m}_p + \kappa C_\perp)\boldsymbol{\nu}^T + K'\boldsymbol{\nu}_\perp \mid K' \in \mathbb{R}^{\mathrm{n}\times(\mathrm{n}-2)} \text{ and } \kappa \in \mathbb{R}\right\}$.

Therefore, the sets $\mathcal{A}^c_{\mathbf{xf}}$, $\mathcal{A}^s_{\mathbf{xf}}$, $\mathcal{A}^o_{\mathbf{xf}}$, $\mathcal{A}^d_{\mathbf{xf}}$, $\mathcal{A}^l_{\mathbf{xf}}$, and $\mathcal{A}^i_{\mathbf{xf}}$ can be easily obtained from Lemmas 3.2.1-3.2.3, as below:

Theorem 3.2.1. *Let* $\mathbf{x} \in \mathbb{R}^{\mathrm{n}}\backslash\{\mathbf{0}\}$, $\mathbf{f} \in \mathbb{R}^{\mathrm{n}}$, *and* $B, C^T \in \mathbb{R}^{\mathrm{n}\times(\mathrm{n}-1)}$ *of full column rank. Then*

1) $$\mathcal{A}^c_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}}\backslash\mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}} & \text{if } \mathbf{x} \notin span\{B\}; \\ \mathcal{A}_{\mathbf{xf}} & \text{if } \mathbf{x} \in span\{B\} \text{ and } \mathbf{f} \notin span\{B\}; \\ \mathcal{A}_{\mathbf{xf}}\backslash\mathcal{A}^{\bar{c}(2)}_{\mathbf{xf}} & \text{if } \mathbf{x}, \mathbf{f} \in span\{B\}, \end{cases}$$

where $\mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}} := \left\{A^{\bar{c}(1)}_p + \tilde{B}K'\mathbf{x}_\perp \mid K' \in \mathbb{R}^{(\mathrm{n}-1)\times(\mathrm{n}-1)} \;\&\; A^{\bar{c}(1)}_p = \frac{\mathbf{f}\mathbf{x}^{\mathrm{T}}}{||\mathbf{x}||^2} - \frac{B_\perp \mathbf{f}}{||\mathbf{x}||^2}B^T_\perp(\mathbf{x}_\perp\tilde{B})^{-1}(\mathbf{x}^T\tilde{B})\mathbf{x}_\perp\right\}$ *is a* $(n-1)^2$*- dimension linear variety,* $\mathcal{A}^{\bar{c}(2)}_{\mathbf{xf}} := \left\{A_p + (\kappa B^T_\perp\boldsymbol{\eta}^T + \tilde{B}K')\mathbf{x}_\perp \mid \tilde{B}^T\mathbf{x}^T_\perp\boldsymbol{\eta} = \mathbf{0},\ K' \in \mathbb{R}^{(\mathrm{n}-1)\times(\mathrm{n}-1)} \;\&\; \kappa \in \mathbb{R}\right\}$ *is a* (n^2-2n+2)*- dimension linear variety, both* $\mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}}$ *and* $\mathcal{A}^{\bar{c}(2)}_{\mathbf{xf}}$ *are in* $\mathcal{A}_{\mathbf{xf}}$ *and in which* (A, B) *is uncontrollable.*

2) $$\mathcal{A}^s_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} & \text{if ``}\mathbf{x} \notin span\{B\} \;\&\; \psi < 0\text{'' or} \\ & \text{``}\mathbf{x} \in span\{B\} \;\&\; \mathbf{f} \notin span\{B\};\text{''} \\ \mathcal{A}_{\mathbf{xf}}\backslash\mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}} & \text{if } \mathbf{x} \notin span\{B\} \;\&\; \psi \geq 0; \\ \mathcal{A}_{\mathbf{xf}}\backslash\mathcal{A}^{\bar{s}}_{\mathbf{xf}} & \text{if } \mathbf{x}, \mathbf{f} \in span\{B\}, \end{cases}$$

where $\psi = (B_\perp\mathbf{f})\left[(B_\perp\mathbf{x}) - (\mathbf{x}_\perp\tilde{B})^{-1}(\mathbf{x}^T\tilde{B})(\mathbf{x}_\perp B^T_\perp)\right]$ *and* $\mathcal{A}^{\bar{s}}_{\mathbf{xf}} := \left\{A \in \mathcal{A}^{\bar{c}(2)}_{\mathbf{xf}} \mid sign(\boldsymbol{\eta}^T\mathbf{x}_\perp B^T_\perp)\cdot\kappa \geq 0\right\}$ *is half of* $\mathcal{A}^{\bar{c}(2)}_{\mathbf{xf}}$ *and in which* (A, B) *is unstabilizable.*

3) $$\mathcal{A}^o_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}}\backslash\mathcal{A}^{\bar{o}}_{\mathbf{xf}} & \text{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \text{if } C\mathbf{x} = 0 \;\&\; \{\mathbf{x}, \mathbf{f}\} \text{ are LI}; \\ \emptyset & \text{if } C\mathbf{x} = 0 \;\&\; \{\mathbf{x}, \mathbf{f}\} \text{ are LD}, \end{cases}$$

where $\mathcal{A}^{\bar{o}}_{\mathbf{xf}} := \left\{A^{\bar{o}}_p + \left[\kappa\frac{C_\perp(\mathbf{x}_\perp C_\perp)^T}{||\mathbf{x}_\perp C_\perp||^2} + K'(\mathbf{x}_\perp C_\perp)_\perp\right]\cdot\mathbf{x}_\perp \mid \kappa \in \mathbb{R},\ \mathrm{K}' \in \mathbb{R}^{\mathrm{n}\times(\mathrm{n}-2)},\ \mathrm{A}^{\bar{o}}_{\mathrm{p}} = \frac{\mathbf{m}_{\mathrm{p}}(\mathbf{x}_\perp \mathrm{C}_\perp)^{\mathrm{T}}}{||\mathbf{x}_\perp\mathrm{C}_\perp||^2}\mathbf{x}_\perp + \frac{\mathbf{f}\mathbf{x}^{\mathrm{T}}}{||\mathbf{x}||^2} \;\&\; \tilde{\mathrm{C}}\mathbf{m}_{\mathrm{p}} = -\frac{\mathbf{x}^{\mathrm{T}}\mathrm{C}_\perp}{||\mathbf{x}||^2}(\tilde{\mathrm{C}}\mathbf{f})\right\}$ *is a* $(n-1)^2$*-dimension linear variety in* $\mathcal{A}_{\mathbf{xf}}$ *and in which* (A, C) *is unobservable.*

4) $\mathcal{A}^{d}_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}^{\bar{d}}_{\mathbf{xf}} & \textit{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \textit{if "}C\mathbf{x} = 0 \ \& \ \{\mathbf{x}, \mathbf{f}\} \textit{ are LI" or} \\ & \textit{"}C\mathbf{x} = 0 \ \& \ \mathbf{f} = \mu\mathbf{x},\ \mu < 0;\textit{"} \\ \emptyset & \textit{if } C\mathbf{x} = 0 \ \& \ \mathbf{f} = \mu\mathbf{x},\ \mu \geq 0, \end{cases}$

where $\mathcal{A}^{\bar{d}}_{\mathbf{xf}} := \left\{ A \in \mathcal{A}^{\bar{o}}_{\mathbf{xf}} \mid \kappa \geq -\frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} - C_\perp^T \mathbf{m}_p \right\}$ *is a half of* $\mathcal{A}^{\bar{o}}_{\mathbf{xf}}$ *and in which* (A, C) *is undetectable.*

5) $\mathcal{A}^{l}_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}^{\bar{l}}_{\mathbf{xf}} & \textit{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \textit{if "}C\mathbf{x} = 0 \ \& \ \{\mathbf{x}, \mathbf{f}\} \textit{ are LI" or} \\ & \textit{"}C\mathbf{x} = 0 \ \& \ \mathbf{f} = \mu\mathbf{x},\ \mu > 0;\textit{"} \\ \emptyset & \textit{if } C\mathbf{x} = 0 \ \& \ \mathbf{f} = \mu\mathbf{x},\ \mu \leq 0, \end{cases}$

where $\mathcal{A}^{\bar{l}}_{\mathbf{xf}} := \left\{ A \in \mathcal{A}^{\bar{o}}_{\mathbf{xf}} \mid \kappa \leq -\frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} - C_\perp^T \mathbf{m}_p \right\}$ *is a half of* $\mathcal{A}^{\bar{o}}_{\mathbf{xf}}$ *and in which* (A, C) *has an unobservable mode on the LHP and* $j\omega$*-axis.*

6) $\mathcal{A}^{i}_{\mathbf{xf}} = \begin{cases} \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}^{\bar{i}}_{\mathbf{xf}} & \textit{if } C\mathbf{x} \neq 0; \\ \mathcal{A}_{\mathbf{xf}} & \textit{if "}C\mathbf{x} = 0 \ \& \ \mathbf{f} \neq \mathbf{0};\textit{"} \\ \emptyset & \textit{if } C\mathbf{x} = 0 \ \& \ \mathbf{f} = \mathbf{0}, \end{cases}$

where $\mathcal{A}^{\bar{i}}_{\mathbf{xf}} := \left\{ A \in \mathcal{A}^{\bar{o}}_{\mathbf{xf}} \mid \kappa = -\frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} - C_\perp^T \mathbf{m}_p \right\}$ *is a subspace of* $(n^2 - 2n)$*-dimension in* $\mathcal{A}^{\bar{o}}_{\mathbf{xf}}$ *and in which* (A, C) *has an unobservable mode on the* $j\omega$*-axis.*

Proof. First, we derive $\mathcal{A}^{c}_{\mathbf{xf}}$ and $\mathcal{A}^{s}_{\mathbf{xf}}$. Let $M_B = [B_\perp^T \ \vdots \ \tilde{B}]$. It is clear that $\bar{A} = M_B^T A M_B$ and $\bar{B} = M_B^T B$ are in the form described in Lemma 3.1.1. By direct calculation, $\bar{A}_{12} = \frac{B_\perp \mathbf{f}}{||\mathbf{x}||^2}(\mathbf{x}^T \tilde{B}) + (B_\perp K)(\mathbf{x}_\perp \tilde{B})$ and $\bar{A}_{11} = \frac{(B_\perp \mathbf{f})(B_\perp \mathbf{x})}{||\mathbf{x}||^2} + (B_\perp K)(\mathbf{x}_\perp B_\perp^T)$. From Theorem 4 [153] and Lemma 3.1.1, (A, B) is uncontrollable $\Leftrightarrow \bar{A}_{12} = 0$; and (A, B) is unstabilizable $\Leftrightarrow \bar{A}_{12} = 0$ and $\bar{A}_{11} \geq 0$. By Lemma 3.2.1, $\mathbf{x}_\perp \tilde{B}$ is singular $\Leftrightarrow \mathbf{x} \in \mathrm{span}\{B\}$. If $\mathbf{x} \notin \mathrm{span}\{B\}$ (i.e., $\mathbf{x}_\perp \tilde{B}$ is nonsingular), then those of K such that $\bar{A}_{12} = 0$ can be parameterized via Lemma 3.2.2 with $(B, M, \boldsymbol{\xi})$ being replaced by $\left(\tilde{B}, \mathbf{x}_\perp \tilde{B}, -\frac{(B_\perp \mathbf{f})}{||\mathbf{x}||^2}(\tilde{B}^T \mathbf{x})\right)$. Combining the parameterization of K with the expression of $\mathcal{A}_{\mathbf{xf}}$ gives the set $\mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}}$. Consequently, $\mathcal{A}^{c}_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}}$. Besides, within $\mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}}$ (i.e., $\bar{A}_{12} = 0$), K satisfies the relation $B_\perp K = -\frac{(B_\perp \mathbf{f})(\mathbf{x}^T \tilde{B})}{||\mathbf{x}||^2}(\mathbf{x}_\perp \tilde{B})^{-1}$. Inserting this relation into $\bar{A}_{11}$ yields $\bar{A}_{11} = \frac{\psi}{||\mathbf{x}||^2}$. Thus, $\bar{A}_{11} < 0 \Leftrightarrow \psi < 0$. Therefore, $\mathcal{A}^{s}_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$ if $\mathbf{x} \notin \mathrm{span}\{B\}$ and $\psi < 0$; and $\mathcal{A}^{s}_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}^{\bar{c}(1)}_{\mathbf{xf}}$ if $\mathbf{x} \notin \mathrm{span}\{B\}$ and $\psi \geq 0$. We now consider the case that $\mathbf{x}_\perp \tilde{B}$ is singular (i.e., $\mathbf{x} \in \mathrm{span}\{B\}$). If $\mathbf{f} \notin \mathrm{span}\{B\}$ (i.e., $B_\perp \mathbf{f} \neq 0$), since $\left[\frac{(B_\perp \mathbf{f})}{||\mathbf{x}||^2}\mathbf{x}^T + (B_\perp K)\mathbf{x}_\perp\right] \notin B_\perp$, we have $\bar{A}_{12} = \left[\frac{(B_\perp \mathbf{f})}{||\mathbf{x}||^2}\mathbf{x}^T + (B_\perp K)\mathbf{x}_\perp\right]\tilde{B} \neq 0$ and thus $\mathcal{A}^{c}_{\mathbf{xf}} = \mathcal{A}^{s}_{\mathbf{xf}} = \mathcal{A}_{\mathbf{xf}}$. When

$\mathbf{f} \in \text{span}\{B\}$ (i.e., $B_\perp \mathbf{f} = 0$), then those of K such that $\bar{A}_{12} = 0$ can be parameterized via Lemma 3.2.2 with $(B, M, \boldsymbol{\xi})$ being replaced by $\big(\tilde{B}, \mathbf{x}_\perp \tilde{B}, \mathbf{0}\big)$. Combining the parameterization of K with the expression of $\mathcal{A}_{\mathbf{xf}}$ gives the set $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$. Consequently, $\mathcal{A}_{\mathbf{xf}}^{c} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$. Besides, within $\mathcal{A}_{\mathbf{xf}}^{\bar{c}(2)}$ (i.e., $\bar{A}_{12} = 0$) and via the parametrization of K, we have $\bar{A}_{11} = \kappa \boldsymbol{\eta}^T \mathbf{x}_\perp B_\perp^T \geq 0 \Leftrightarrow \text{sign}(\boldsymbol{\eta}^T \mathbf{x}_\perp B_\perp^T) \cdot \kappa \geq 0$ and thus $\mathcal{A}_{\mathbf{xf}}^{s} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}_{\mathbf{xf}}^{\bar{s}}$.

Next, we study $\mathcal{A}_{\mathbf{xf}}^{o}$, $\mathcal{A}_{\mathbf{xf}}^{d}$, and $\mathcal{A}_{\mathbf{xf}}^{i}$ (the proof of $\mathcal{A}_{\mathbf{xf}}^{l}$ is quite similar to $\mathcal{A}_{\mathbf{xf}}^{d}$ and thus omitted). Let $M_C = [C_\perp \vdots \tilde{C}^T]$. Then $\bar{C} = CM_C$ has the form of Corollary 3.1.1. By direct calculation, $\bar{A}_{21} = \frac{\mathbf{x}^T C_\perp}{||\mathbf{x}||^2}(\tilde{C}\mathbf{f}) + (\tilde{C}K)(\mathbf{x}_\perp C_\perp)$ and $\bar{A}_{11} = \frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} + (C_\perp^T K)(\mathbf{x}_\perp C_\perp)$. If $\mathbf{x}_\perp C_\perp \neq 0$ (i.e., $\tilde{C}\mathbf{x} \neq 0$), then the set of K such that $\bar{A}_{21} = 0$ can be parameterized via Lemma 3.2.3 with $(C, \boldsymbol{\nu}, \boldsymbol{\xi}) = \left(\tilde{C}, \mathbf{x}_\perp C_\perp, -\frac{\mathbf{x}^T C_\perp}{||\mathbf{x}||^2}(\tilde{C}\mathbf{f})\right)$. Inserting the parameterization of K into the expression of $\mathcal{A}_{\mathbf{xf}}$ gives the set $\mathcal{A}_{\mathbf{xf}}^{\bar{o}}$. Consequently, $\mathcal{A}_{\mathbf{xf}}^{o} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}_{\mathbf{xf}}^{\bar{o}}$. Besides, within $\mathcal{A}_{\mathbf{xf}}^{\bar{o}}$ and via the parametrization of K, $\bar{A}_{11} = \frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} + C_\perp^T \mathbf{m}_p + \kappa \geq 0 \Leftrightarrow \kappa \geq -\frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} - C_\perp^T \mathbf{m}_p$. Thus, we have the above expression of $\mathcal{A}_{\mathbf{xf}}^{\bar{d}}$ and $\mathcal{A}_{\mathbf{xf}}^{d} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}_{\mathbf{xf}}^{\bar{d}}$. Similarly, $\mathcal{A}_{\mathbf{xf}}^{\bar{i}}$ can be obtained. We now consider the case of $\mathbf{x}_\perp C_\perp = 0$ (i.e., $\tilde{C}\mathbf{x} = 0$), which implies that $\mathbf{x}^T C_\perp \neq 0$, and $\bar{A}_{21} = 0 \Leftrightarrow \tilde{C}\mathbf{f} = 0 \Leftrightarrow \{\mathbf{x}, \mathbf{f}, C_\perp\}$ are collinear. As a result, $\mathcal{A}_{\mathbf{xf}}^{o} = \mathcal{A}_{\mathbf{xf}}^{d} = \mathcal{A}_{\mathbf{xf}}^{i} = \mathcal{A}_{\mathbf{xf}}$ if $\{\mathbf{x}, \mathbf{f}\}$ are LI; and $\mathcal{A}_{\mathbf{xf}}^{o} = \emptyset$ if $\{\mathbf{x}, \mathbf{f}\}$ are LD. When $\{\mathbf{x}, \mathbf{f}\}$ are LD with the relation $\mathbf{f} = \mu\mathbf{x}$ and $\mu \geq 0$, then we have $\bar{A}_{11} = \frac{(C_\perp^T \mathbf{f})(\mathbf{x}^T C_\perp)}{||\mathbf{x}||^2} = \mu\frac{(\mathbf{x}^T C_\perp)^2}{||\mathbf{x}||^2} \geq 0$ and thus $\mathcal{A}_{\mathbf{xf}}^{d} = \emptyset$. On the contrary, if $\mathbf{f} = \mu\mathbf{x}$, $\mu < 0$, then $\bar{A}_{11} = \mu\frac{(\mathbf{x}^T C_\perp)^2}{||\mathbf{x}||^2} < 0$ and $\mathcal{A}_{\mathbf{xf}}^{d} = \mathcal{A}_{\mathbf{xf}}$. In addition, because $C\mathbf{x} = 0$ and $\{\mathbf{x}, \mathbf{f}\}$ are LD with $\mathbf{f} = \mu\mathbf{x}$, we have $\mathcal{A}_{\mathbf{xf}}^{i} = \mathcal{A}_{\mathbf{xf}} \Leftrightarrow \bar{A}_{11} = \mu\frac{(\mathbf{x}^T C_\perp)^2}{||\mathbf{x}||^2} \neq 0 \Leftrightarrow \mu \neq 0 \Leftrightarrow \mathbf{f} \neq \mathbf{0}$. □

It is interesting to note from Theorem 3.2.1 that the sets $\mathcal{A}_{\mathbf{xf}}^{c}$ and $\mathcal{A}_{\mathbf{xf}}^{s}$ are always non-empty as long as $\text{rank}(B) \geq n-1$, and if restricting to the planar case, then such requirement could be weakened to $B \neq 0$. Note that the results of Theorem 3.2.1 agree with Theorem 2.2.1.

Then, we consider the remaining case of $\text{rank}(B) = p$, $1 \leq p < n-1$, while the derivation for $1 \leq \text{rank}(C) < n-1$ can be obtained similarly. Transform the coordinate as give by M_B (3.2), we have $\bar{A} = M_B^T A M_B$, $\bar{A}_{12} = \frac{B_\perp \mathbf{f}}{||\mathbf{x}||^2}(\mathbf{x}^T \tilde{B}) + (B_\perp K)(\mathbf{x}_\perp \tilde{B})$ and $\bar{A}_{11} = \frac{(B_\perp \mathbf{f})(B_\perp \mathbf{x})}{||\mathbf{x}||^2} + (B_\perp K)(\mathbf{x}_\perp B_\perp^T)$. Thus we have the following Theorem 3.2.2, whose proof is straightforward.

Theorem 3.2.2.

1) $\mathcal{A}^c_{\mathbf{xf}} = \left\{ A_p + K\mathbf{x}_\perp \mid rank \left[\lambda I_{n-p} - \bar{A}_{11} \vdots \bar{A}_{12} \right] = n - p, \forall \lambda \in \mathrm{C} \right\}.$

2) $\mathcal{A}^s_{\mathbf{xf}} = \left\{ A_p + K\mathbf{x}_\perp \mid rank \left[\lambda I_{n-p} - \bar{A}_{11} \vdots \bar{A}_{12} \right] = n - p, \forall \lambda \in \sigma(\bar{A}_{11}) \right.$
$\left. \text{with } Re(\lambda) \geq 0 \right\}.$

Theorem 3.2.2 presents the analytical result and, via the QR scheme for example, can be improved for its numerical performance. After having the sets $\mathcal{A}_{\mathbf{xf}}$, $\mathcal{A}^s_{\mathbf{xf}}$, $\mathcal{A}^l_{\mathbf{xf}}$ and $\mathcal{A}^i_{\mathbf{xf}}$ presented above, the solutions $\mathcal{A}^{sl}_{\mathbf{xf}}$ and $\mathcal{A}^{si}_{\mathbf{xf}}$ can be easily obtained according to the ranks of B and C. Note that, [153] includes the planar case for an easy reference.

3.3 Concluding Remarks

In consistence with the necessary and sufficient conditions for the solvability of SDRE/SDDRE in Chapter 2, in this chapter, a representation for all feasible SDC matrices is given, together with an efficient method to easily determine the solutions' property. However, how to schematically correlate the feasible SDC matrices with the system's performance (such as stability, optimality, robustness, and reliability) needs further investigation. In the next chapter, the connections to various research topics related to the SDRE/SDDRE scheme are going to be studied, which might attract the interest of the research on both schemes among the control community.

Chapter 4

Connections, Potentials, and Applications [1]

Based on the proposed results in Chapters 2-3, in this chapter we discuss several connections to the literature, to either further the studies or open a new avenue, with respect to the SDRE/SDDRE scheme:

4.1 Domain of Attraction (DOA)

In [34, 87], the authors have described such procedure for the estimation of DOA via SDRE scheme using vector norms, but without guidelines on the construction of SDC matrices when the SDRE solvability condition is violated. Nonetheless, combining results of Theorem 2.2.1 and Algorithm 2.3.1 could implement successfully the Procedure (1) of DOA estimation [87]. In addition, Theorem 3.2.1 has provided all the feasible SDC matrices, and therefore the overall DOA estimation is the largest of all estimations by applying every SDC matrix in $\mathcal{A}^{si}_{\mathbf{xf}}$, $\mathcal{A}^{sl}_{\mathbf{xf}}$, or others. For example, the SDC matrix in Example 2 of [87] can be parameterized with $K = \left(\frac{-2x_1^3 - x_2}{||\mathbf{x}||}, \frac{x_1}{||\mathbf{x}||}\right)$, and is included in the non-singleton $\mathcal{A}^{so}_{\mathbf{xf}}$.

Alternatively, as compared to the vector-norm method [87], [34] proposed a less conservative improvement to estimate the DOA via SDRE, suitable for both autonomous and controlled nonlinear systems. For the latter case, the

[1] Journal version at [159, 160]

procedure mainly focuses on the closed-loop matrix $A_{CL}(\mathbf{x})$. For this procedure, an appropriate Lyapunov function $V(\mathbf{x})$ is required to be determined by tuning the state-weighting matrix $Q(\mathbf{x})$, which is obtained from the linearized system (with asymptotic stability) around the origin. Thus the largest level set of $V(\mathbf{x})$, completely inside the region where $\dot{V}(\mathbf{x}) < 0$, defines a lower bound of the DOA [34]. Note that, if several Lyapunov functions are suitable, then the estimation of the DOA is just the union of the estimates corresponding to those Lyapunov functions. Interestingly, such a procedure could also be connected with the proposed results for possibly larger estimation, by fully exploiting the design flexibility/degree of freedom of the SDRE scheme. Either by Algorithm 2.3.1 to easily generate a feasible SDC matrix or Theorems 3.2.1-3.2.2 for the other candidates, the resulting SDRE (1.19) is solvable while the closed-loop matrix $A_{CL}(\mathbf{x})$ is Hurwitz, as guaranteed by Theorem 2.2.1. All the resulting $A_{CL}(\mathbf{x})$ serve as appropriate candidates as needed in Eq. (17) or Procedure (1) of [34], and then could be fed into the procedure of estimation (Section III.B in [34]). Therefore, the estimation of the DOA is just the union of all the estimates corresponding to those different closed-loop matrices $A_{CL}(\mathbf{x})$, originating from SDC matrices in $\mathcal{A}^{si}_{\mathbf{xf}}$, $\mathcal{A}^{sl}_{\mathbf{xf}}$, or others.

To sum up, the following Algorithms 4.1.1-4.1.2 are summarized to describe the estimation of DOA via both the above-mentioned schemes. Note that, an intuitive assumption is that the origin is an equilibrium point. Moreover, among those SDC sets ($\mathcal{A}^{si}_{\mathbf{xf}}$, $\mathcal{A}^{sl}_{\mathbf{xf}}$, or others), searching in $\mathcal{A}^{si}_{\mathbf{xf}}$ thoroughly would yield the largest DOA estimation, since $\mathcal{A}^{s\alpha}_{\mathbf{xf}} \subseteq \mathcal{A}^{si}_{\mathbf{xf}},\ \alpha = l, o$, and any SDC matrix in $\mathcal{A}^{si}_{\mathbf{xf}}$ would result the corresponding SDRE (1.19) with an unique, symmetric, positive semi-definite, and stabilizing solution, as guaranteed by Theorem 2.2.1.

Algorithm 4.1.1: Estimation of DOA via SDRE using Vector Norms

input : $\mathbf{x}, \mathbf{f}, B, C$, and R

output : D, the estimation domain of DOA

1 j=1.

`/*` $\mathcal{A}^{si}_{\mathbf{xf}}$ `would yield the largest DOA */`

2 **while** $j <$ card$(\mathcal{A}^{si}_{\mathbf{xf}})$, the cardinality of the set $\mathcal{A}^{si}_{\mathbf{xf}}$ **do**

3 Choose a domain S, where $\mathbf{0} \in S$.

4 Represent A_{CL} in terms of the state vector $\mathbf{x}$ and feedback gains $k_1(\mathbf{x}), \cdots, k_n(\mathbf{x})$, as in Eq. (1.20) but regarding the time-invariant systems, where $[k_1(\mathbf{x}), \cdots, k_n(\mathbf{x})]^T = K(\mathbf{x})$.

5 Calculate $\inf_{\mathbf{x}\in S} k_i(\mathbf{x})$ and $\sup_{\mathbf{x}\in S} k_i(\mathbf{x})$, $i = 1, \cdots, n$.

6 Construct an overvaluing matrix M (Section 2 in [87]) based on the information of Steps 3-5.

7 **if** M is not Hurwitz **then**

8 either “break;”, or “go to Step 3;”;

9 $D_j = D_1 \cup D_c \cup D_\infty$, where D_1, D_c, and D_∞ is given by Theorem 1 in [87].

10 Choose another untried $A \in \mathcal{A}^{si}_{\mathbf{xf}}$, parameterized by Algorithm 2.3.1 or Theorems 3.2.1-3.2.2.

11 j=j+1.

12 $D = \cup D_j$.

Algorithm 4.1.2: Estimation of DOA via SDRE using Lyapunov Analysis

input : $\mathbf{x}, \mathbf{f}, B, C$, and R

output : D, the estimation domain of DOA

```
j=1.
/* 𝒜_xf^si would yield the largest DOA */
while j < card(𝒜_xf^si), the cardinality of the set 𝒜_xf^si do
```

j=1.

/* $\mathcal{A}^{si}_{\mathbf{xf}}$ would yield the largest DOA */

while $j < \mathrm{card}(\mathcal{A}^{si}_{\mathbf{xf}})$, the cardinality of the set $\mathcal{A}^{si}_{\mathbf{xf}}$ **do**

- Denote $A^{\mathbf{0}}_{CL} = A_{CL}(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}$ and $Q^{\mathbf{0}} = Q(\mathbf{x})\big|_{\mathbf{x}=\mathbf{0}}$.
- By tuning $Q^{\mathbf{0}}$, solve the Lyapunov function

$$A^{\mathbf{0}}_{CL} P + P A^{\mathbf{0}}_{CL} = -Q^{\mathbf{0}},$$

  such that a Lyapunov function, $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, is determined.
- $L = \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{n}} \middle| \dot{V}(\mathbf{x}) < 0 \right\}$.
- $\underline{V} = \inf\limits_{\mathbf{x} \in \mathbb{R}^{\mathrm{n}} \setminus \mathrm{L}} V(\mathbf{x})$.
- $\bar{V} = \sup\limits_{\mathbf{x} \in \mathbb{R}^{\mathrm{n}}} \{ V(\mathbf{x}) | V(\mathbf{x}) < \underline{V} \}$.
- $D_j = \left\{ \mathbf{x} \in \mathbb{R}^{\mathrm{n}} \middle| V(\mathbf{x}) \le \bar{V} \right\}$, a simply connected domain.
- Choose another untried $A \in \mathcal{A}^{si}_{\mathbf{xf}}$, parameterized by Algorithm 2.3.1 or Theorems 3.2.1-3.2.2.
- j=j+1.

$D = \cup D_j$.

4.2 Computational Performance of Solving SDRE

In real-time applications of the SDRE scheme, especially for high-dimension systems (e.g. the fully embedded SDRE controller of twelve states for real-time control on an autonomous helicopter [29]), the main computational load lies in solving SDRE (1.19) repeatedly at each sampling instant [51]. Some works have been done for alleviating the computational burden (see [90] and the references therein). Among these, it is worth mentioning that [223] and [95] are of great importance. [95] has presented an efficient algorithm (called modified Newton method, MNM) for solving SDRE (1.19), which seems to be better than the MATLAB® built-in solver by Schur algorithm [11], since the latter operates on a $2n \times 2n$ Hamiltonian matrix for an $n-$dim Riccati equation [51]. The computational time of MNM strictly depends on the design degree of freedom, i.e. the maximum error allowed, but requires an initial guess of the stabilizing solution in the very first beginning. Thanks to [223], such an initial guess can be readily found. However, when the SDRE scheme on system (1.14) is equipped with both results [95, 223], the following two assumptions are inherited: (1) the pair (A, B) is stabilizable, i.e. the set $\mathcal{A}^s_{\mathbf{xf}}$ is nonempty; (2) there exists a symmetric solution of the following Riccati inequality at every $(\mathbf{x}, t)$, $\mathbf{x} \neq \mathbf{0}$:

$$\begin{aligned} &A^T(\mathbf{x},t)P(\mathbf{x},t) + P(\mathbf{x},t)A(\mathbf{x},t) + Q(\mathbf{x},t) \\ &-P(\mathbf{x},t)B(\mathbf{x},t)R^{-1}(\mathbf{x},t)B^T(\mathbf{x},t)P(\mathbf{x},t) \geq 0. \end{aligned} \tag{4.1}$$

Since $\mathcal{A}^{s\alpha}_{\mathbf{xf}} \subseteq \mathcal{A}^{si}_{\mathbf{xf}} \subseteq \mathcal{A}^{s}_{\mathbf{xf}}$, $\alpha = l, o$, both assumptions could be removed by the results of Chapters 2-3, therefore the MNM starting with the initial guess [223] can be used to solve the SDRE (1.19) for tunable and efficient computational performance, briefed as the following algorithm and demonstrated via MATLAB® in Ex. 5.1.3:

Algorithm 4.2.1: Alternative SDRE (1.19) solver for tunable computational performance

input : A, B, Q, R and ϵ

output : P_{k+1}

`/* Obtain` P_0`, an initial guess for Hurwitz` A_{CL} `[223] */`

Transform A into real Schur form as $\tilde{A} = U^T A U$ (MATLAB® function "schur").

Select $0 < \beta < ||A||$.

Compute $\tilde{B} = U^T B$ and $\tilde{C} = \tilde{B}\tilde{B}$.

Solve the reduced-form Lyapunov function for $\tilde{Z}$ (MATLAB® function "lyap")

$$\left(\tilde{A} + \beta I_n\right) \tilde{Z} + \tilde{Z} \left(\tilde{A} + \beta I_n\right)^T = 2\tilde{C}. \tag{4.2}$$

Solve $\tilde{Z} X = \tilde{B}$.

Compute $P_0 = X^T U^T$.

`/* Modified Newton method (MNM) solver [95] */`

$k = 0$.

Denote $\mathcal{R}'_X(S) = -\left[S(A - BR^{-1}B^T X) + (A - BR^{-1}B^T X)^T S\right]$ and $\mathcal{R}(X) = XBR^{-1}B^T X - XA - A^T X - Q$.

while $k > 0$ **do**

    Solve $\mathcal{R}'_{P_k}(H) = \mathcal{R}(P_k)$ for H (MATLAB® function "lyap");

    $P_{k+1} = P_k - 2H$;

    if $\mathcal{R}(P_{k+1}) < \epsilon$ **then**

        break;

    $P_{k+1} = P_k - H$;

    if $\mathcal{R}(P_{k+1}) < \epsilon$ **then**

        break;

    $k = k + 1$;

4.3 Tracking (Command Following)

In [12, 51, 64, 66, 67], the authors have implemented the SDRE controller as an integral servomechanism for the tracking problem, which is described as follows. Decompose the state $\mathbf{x} = [\mathbf{x}_R^T, \mathbf{x}_N^T]^T$, where $\mathbf{x}_R \in \mathbb{R}^\gamma$ is desired to track the command signal $\mathbf{r} \in \mathbb{R}^\gamma$. Let $\mathbf{x}_I$ be the integral states of $\mathbf{x}_R$, and consider the following augmented time-invariant system (noting that in this case $\mathbf{f}, A, B, Q$ and R depend only on the state):

$$\dot{\tilde{\mathbf{x}}} = \tilde{A}(\tilde{\mathbf{x}})\tilde{\mathbf{x}} + \tilde{B}(\tilde{\mathbf{x}})\mathbf{u}_{track}, \;\; \tilde{\mathbf{x}} = [\mathbf{x}_I^T, \mathbf{x}_R^T, \mathbf{x}_N^T]^T,$$

$$\tilde{A}(\tilde{\mathbf{x}}) = \left[\begin{array}{c:cc} 0 & I_\gamma & 0 \\ \hdashline 0 & \multicolumn{2}{c}{A(\mathbf{x})} \end{array}\right], \;\; \tilde{B}(\tilde{\mathbf{x}}) = \begin{bmatrix} 0 \\ B(\mathbf{x}) \end{bmatrix}, \tag{4.3}$$

with the performance index (1.18) but replaced by $\tilde{\mathbf{x}}$ and $\tilde{Q}(\tilde{\mathbf{x}}) = \tilde{C}^T(\tilde{\mathbf{x}})\tilde{C}(\tilde{\mathbf{x}})$. The SDRE integral servo controller is given by:

$$\mathbf{u}_{track} = -R^{-1}(\mathbf{x})\tilde{B}^T(\tilde{\mathbf{x}})\tilde{P}(\tilde{\mathbf{x}}) \begin{bmatrix} \mathbf{x}_I - \int \mathbf{r}\, dt \\ \mathbf{x}_R - \mathbf{r} \\ \mathbf{x}_N \end{bmatrix}, \tag{4.4}$$

where $\tilde{P}(\tilde{\mathbf{x}})$ is solved from the corresponding SDRE with $\tilde{A}(\tilde{\mathbf{x}}), \tilde{B}(\tilde{\mathbf{x}}), \tilde{Q}(\tilde{\mathbf{x}})$, and $R(\mathbf{x})$ for the augmented system.

For brevity, denote $\tilde{A} = \tilde{A}(\tilde{\mathbf{x}}), \tilde{B} = \tilde{B}(\tilde{\mathbf{x}})$, and $\tilde{C} = \tilde{C}(\tilde{\mathbf{x}})$. In view of Lemma 3.1.1, we treat the pair $(0, [I_\gamma \vdots 0])$ in $\tilde{A}$ as $(\bar{A}_{11}, \bar{A}_{12})$. Since I_γ provides γ LI columns, we have that $(\tilde{A}, \tilde{B})$ is controllable (resp., stabilizable) $\Leftrightarrow$ (A, B) is controllable (resp., stabilizable). Similarly, in view of Corollary 3.1.1, we at first first consider the case rank$(C) = q \geq \gamma$ and let $\tilde{C} = [C_I, C]$ with $C_I := C_I(\tilde{\mathbf{x}}) \in \mathbb{R}^{q\times\gamma}$ being full column rank. Then $(\tilde{A}, \tilde{C})$ is observable $\Leftrightarrow$ (A, C) is observable, and the relation holds for the other cases of detectability and no unobservable mode on the $j\omega$-axis. On the other hand, the case for $q < \gamma$ would result that $(\tilde{A}, \tilde{C})$ has an unobservable mode at the origin, even if C_I being full row rank. Note that the above discussion generalizes the findings in [12, 51, 66, 67], which requires the diagonal elements of $\tilde{Q}(\tilde{\mathbf{x}})$ corresponding to $\mathbf{x}_I$ to be nonzero.

To conclude, simply by choosing rank$(Q(\mathbf{x})) \geq \gamma$ and C_I to be full column rank for all nonzero states, then the results of Chapters 2-3 directly apply to the augmented system (4.3) for tracking. For illustrations, [67] includes an UAV-tracking design based on the above-mentioned integral servomechanism via SDRE, while [64] applies such a scheme to the tracking control of a continuously stirred tank reactor. Additionally, such a tracking scheme will also be demonstrated in the control of an overhead crane, via MATLAB® in Ex. 5.2.3.

4.4 Closed-Form Solution of SDDRE

Currently, there exists a problem in solving SDDRE (1.16), since the states in the future are not known ahead of time. Therefore the integration backward from t_f to t seems impossible, which could be approximated by a forward integration technique under some assumptions (e.g. controllable factorization of the drift term) [105, 194]. Starting around the 1970s, a significant interest has been focused on the problem of finding numerically reliable and efficient algorithms for the integration of the differential Riccati equation (DRE) [73]. Some representative algorithms are the so-called Bernoulli substitution technique, the Chandrasekhar decomposition, and the modified Anderson-Moore method [187]. However, the former two algorithms mentioned above are not suitable for studying some important properties (e.g. finite escape time, limiting behavior and mechanism of attraction) of the solutions of SDDRE (1.16). The last method requires assumptions of fixing the values of closed-loop matrix and $B(\mathbf{x})$ at current time, so the SDDRE scheme can be approximately implemented [111, 113].

However, to the authors' understanding, [188] is the first to establish closed-form formulae for the solution of DRE but with restrictions, namely Theorems 2-3 in [188], which could be utilized for the presented SDDRE scheme (with the Popov matrix as diag$[Q, R]$). Theorem 3 in [188] considers the case that (1) (A, B) is stabilizable, and (2) the Hamiltonian matrix has no eigenvalues on the $j\omega$-axis. Note that under the condition of (1), the condition of (2) is equivalent with that (A, C) has no eigenvalues on the $j\omega$-axis [261]. By Theorem 2.2.1, such case corresponds to $\mathcal{A}^{si}_{\mathbf{xf}}$, and an element in the non-singleton $\mathcal{A}^{si}_{\mathbf{xf}}$ could be easily obtained by Algorithm 2.3.1. Thus the corresponding SDDRE (1.16) has the following closed-form solution $P(\mathbf{x}, t)$ *pointwisely*: (Note that the definitions of variables adopted from [188] are *point-wisely* identical and for the sake of simplicity, dependence on $\mathbf{x}$ and t in some places are omitted.)

$$
\begin{aligned}
P(\mathbf{x}, t) &= T_s^{-T} \cdot \Lambda \cdot \Pi^{-1} \cdot T_s^{-1}, \text{ with} \qquad (4.5)\\
\Pi(\mathbf{x}, t) &= e^{A_{P+} \cdot t}\Phi + \Omega e^{A^{\star} \cdot (t - t_f)}\Psi
\end{aligned}
$$

$$
\begin{aligned}
\Lambda(\mathbf{x},t) &= P^{+}e^{A_{P^+}\cdot t}\Phi + P^{\star}e^{A^{\star}\cdot(t-t_f)}\Psi \\
\Phi(\mathbf{x},t) &:= e^{-A_{P^+}\cdot t_f}\cdot\left\{\begin{bmatrix} 0 & 0 \\ 0 & \Delta_c^{-1}\end{bmatrix}(T_s^T S T_s - P^{+}) + I_n\right\} \\
\Psi(\mathbf{x},t) &:= \begin{bmatrix} I_{n_u} & -P_{uc}^{+}\Delta_c^{-1} \\ 0 & -\Delta_c^{-1}\end{bmatrix}(T_s^T S T_s - P^{+}),
\end{aligned}
$$

where $T_s(\mathbf{x})$ is a change of basis matrix with its last n_c columns spanning the reachable subspace from the origin (denoted $\mathcal{R}(\mathbf{x})$), $n_c(\mathbf{x}) := \dim(\mathcal{R})$, $n_u(\mathbf{x}) := n - n_c$ and $\Omega(\mathbf{x}) := \mathrm{diag}[0_{n_u\times n_u}, I_{n_c}]$. Moreover, in the new basis transformed by T_s, we have $\tilde{A}(\mathbf{x}) := T_s^{-1}AT_s$, $\tilde{B}(\mathbf{x}) := T_s^{-1}B = [0, B_c^T]^T$, $\tilde{Q}(\mathbf{x}) := T_s^T Q T_s$, $\tilde{P}^{+}(\mathbf{x}) := T_s^T P^{+} T_s$, $\tilde{P}^{-}(\mathbf{x}) := T_s^T P^{-} T_s$, and denote

$$
\tilde{A}(\mathbf{x}) = \begin{bmatrix} A_u & 0 \\ A_{uc} & A_c \end{bmatrix},\ \tilde{Q}(\mathbf{x}) = \begin{bmatrix} Q_u & Q_{uc} \\ Q_{uc}^T & Q_c \end{bmatrix},
$$

$$
\tilde{P}^{+}(\mathbf{x}) = \begin{bmatrix} P_u^{+} & P_{uc}^{+} \\ (P_{uc}^{+})^T & P_c^{+} \end{bmatrix},\ \tilde{P}^{-}(\mathbf{x}) = \begin{bmatrix} P_u^{-} & P_{uc}^{-} \\ (P_{uc}^{-})^T & P_c^{-} \end{bmatrix},
$$

$$
A^{\star}(\mathbf{x}) := \begin{bmatrix} -A_u^T & -Q_{uc} - A_{uc}^T P_c^{-} \\ 0 & A_c - B_c R^{-1} B_c^T P_c^{-} \end{bmatrix},
$$

where $P^{+}(\mathbf{x}) \geq 0$ (resp. $P^{-}(\mathbf{x}) \leq 0$) is the unique, symmetric, and stabilizing (resp. anti-stabilizing) solution of the corresponding SDRE (1.19), $A_{p^+}(\mathbf{x}) := A - BR^{-1}B^T P^{+}$, $P^{\star}(\mathbf{x}) = \mathrm{diag}[I_{n_u}, P_c^{-}]$, and $\Delta_c(\mathbf{x}) := P_c^{+} - P_c^{-} > 0$.

Note that the related computational effort mainly lies in solving the associated SDRE (1.19), which is just within the scope of Section 4.2. Besides, Theorem 2 in [188] alternatively gives an explicit expression for the solution of the DRE under the sign-controllability assumption, and its connection to the SDDRE scheme can be similarly investigated as above.

It is worth mentioning that, we make it more convenient for the user with respect to the contributions [98,110,111,113,194], by removing the assumption made for the solvability of the SDRE/SDDRE (1.19/1.16). Specifically, while the other contributions have an assumption of stabilizability (resp., controllability) and observability made in [110,111,113] (resp., [98,194]), which is difficult to test by the user, this thesis has replaced it by a necessary and sufficient condition in Theorem 2.2.1. Hence a feasible SDC matrix can be easily constructed by Algorithm 2.3.1, and the capability of the SDDRE scheme (such as the global stability [110]) is developing progressively and promisingly.

In the next chapter, the refined SDDRE scheme will be demonstrated via MATLAB® in Ex. 5.1.2.

4.5 Optimality Issue

We consider the time-invariant case of the SDDRE scheme, noting that $\mathbf{f}, A, B, Q$ and R depend only on the state, while the derivation for the SDRE scheme is quite similar and thus omitted. Denote $J^* = J^*(\mathbf{x}, t) = \min\limits_{u} J_{SDDRE}$ and assume that J^* is given, it is well known that the optimal control for the FTHNOC problem can be obtained from the following:

$$u^* = -R^{-1}B^T\left(\frac{\partial J^*}{\partial \mathbf{x}}\right)^T, \tag{4.6}$$

where $\frac{\partial J^*}{\partial \mathbf{x}}$ satisfies the HJB equation:

$$\frac{\partial J^*}{\partial t} + \left(\frac{\partial J^*}{\partial \mathbf{x}}\right)^T \mathbf{f} - \frac{1}{2}\frac{\partial J^*}{\partial \mathbf{x}} BR^{-1}B^T(\frac{\partial J^*}{\partial \mathbf{x}})^T + \frac{1}{2}\mathbf{x}^T Q\mathbf{x} = 0. \tag{4.7}$$

Comparing (1.17) and (4.6), it is obvious that

$$u_{SDDRE} = u^* \Leftrightarrow \Delta \in B^{\perp}, \text{ where } \Delta = \Delta(\mathbf{x}, \mathrm{t}) = \mathrm{P}\mathbf{x} - (\frac{\partial \mathrm{J}^*}{\partial \mathbf{x}})^{\mathrm{T}}. \tag{4.8}$$

Rewrite $\frac{\partial J^*}{\partial \mathbf{x}} = (P\mathbf{x} - \Delta)^T$ and by Lemma 2-22 of [22], we have

$$\begin{aligned}
J^* &= \mathbf{x}^T \int_0^1 \left[P(\xi\mathbf{x}, t)\xi\mathbf{x} - \Delta(\xi\mathbf{x}, t)\right] d\xi \\
&= \mathbf{x}^T \left[\int_0^1 P(\xi\mathbf{x}, t)\xi d\xi\right] \mathbf{x} - \mathbf{x}^T \int_0^1 \Delta(\xi\mathbf{x}, t) d\xi \\
&= \mathbf{x}^T \left[\frac{1}{2}P(\xi\mathbf{x}, t)\xi^2\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{\partial P(\xi\mathbf{x}, t)}{\partial \xi}\xi^2 d\xi\right] \mathbf{x} - \mathbf{x}^T \int_0^1 \Delta(\xi\mathbf{x}, t) d\xi \\
&= \frac{1}{2}\mathbf{x}^T P(\mathbf{x}, t)\mathbf{x} - \frac{1}{2}\mathbf{x}^T \left[\int_0^1 \frac{\partial P(\xi\mathbf{x}, t)}{\partial \xi}\xi^2 d\xi\right] \mathbf{x} - \mathbf{x}^T \int_0^1 \Delta(\xi\mathbf{x}, t) d\xi,
\end{aligned}$$

and thus $\frac{\partial J^*}{\partial t} = \frac{1}{2}\mathbf{x}^T\dot{P}(\mathbf{x},t)\mathbf{x} + \Gamma$, where $\Gamma = \Gamma(\mathbf{x},t)$ denotes the remaining terms.

Therefore, the HJB (4.7) can be reformulated as

$$\begin{aligned} & \frac{1}{2}\mathbf{x}^T\dot{P}\mathbf{x} + \Gamma + (\mathbf{x}^TP - \Delta^T)A\mathbf{x} + \frac{1}{2}\mathbf{x}^TQ\mathbf{x} \\ & = \frac{1}{2}(\mathbf{x}^TP - \Delta^T)BR^{-1}B^T(P\mathbf{x} - \Delta) \\ \Leftrightarrow \quad & \mathbf{x}^T(\dot{P} + A^TP + PA - PBR^{-1}B^TP + Q)\mathbf{x} = 2(\Delta^T\mathbf{f} - \Gamma), \qquad (4.9) \\ & \text{since } \mathbf{x}^TPA\mathbf{x} = (\mathbf{x}^TPA\mathbf{x})^T = \mathbf{x}^TA^TP\mathbf{x}. \end{aligned}$$

Therefore, from Eqs. (4.8)-(4.9), we may summarize the following theorem and remark, including the counterpart for the SDRE scheme:

Theorem 4.5.1. *If $\Delta \in B^{\perp}$ and $\Delta^T\mathbf{f} = \Gamma$ (resp., $\Delta \in [B, \mathbf{f}]^{\perp}$), then the SDDRE (1.16) (resp., SDRE (1.19)) is the corresponding HJB (resp., HJ) equation.*

Remark 1. The SDDRE (resp., SDRE) scheme recovers the optimal control if the corresponding solution of the SDDRE (1.16) (resp., SDRE (1.19)) satisfies the conditions in Theorem 4.5.1. However, how to precisely select the corresponding SDC matrix from all possible candidates via the presented parametrization, is worth investigating and currently under development. As a special case, the optimal choice $A(\mathbf{x}) \in \mathcal{A}^{so}_{\mathbf{xf}}$ in Example 7 of [116] can be parameterized with $K = \left(\frac{-x_1^2x_2 - x_2}{||\mathbf{x}||}, \frac{x_1}{||\mathbf{x}||} - x_1 \cdot ||\mathbf{x}||\right)$. Additionally, in Example 2.1 of [115], $K = \left(\frac{-x_1}{||\mathbf{x}||}, \frac{\frac{1}{2}x_1 - x_2 - \frac{1}{2}x_1g^2(\mathbf{x})}{||\mathbf{x}||}\right)$ correctly parameterizes the optimal SDC matrix $A(\mathbf{x}) \in \mathcal{A}^{co}_{\mathbf{xf}}$.

4.6 Concluding Remarks

In this chapter, several connections of the proposed scheme to the literature are established, which mainly alleviate the design burden in the initial stage of the SDRE/SDDRE approach. Certainly, there are more topics of research related to the SDRE/SDDRE scheme, such as in Table 1.2, and it is promising that the proposed results in Chapters 2-3 could be applied. Such works would definitely act as possible future research, and any advance(s) would contribute to further reducing the leap of faith for the SDRE/SDDRE paradigm.

Besides, we summarize the following Algorithm 4.6.1 for the refined implementation of the SDDRE scheme, as compared to the general procedure briefed in Section 1.2. This is demonstrated via MATLAB® in Ex. 5.1.2. Note that we may simply choose $Q(\mathbf{x}, t)$ to be invertible for all nonzero states and time, such that, by Theorem 2.2.1, $\mathcal{A}^{si}_{\mathbf{xf}} \neq \emptyset$. Similarly, the counterpart for the refined SDRE scheme could also be summarized as the following Algorithm 4.6.2, as compared to the general procedure briefed in Section 1.2. This is demonstrated via MATLAB® in Chapter 5, e.g. Ex. 5.1.3. Note that we may simply choose $Q(\mathbf{x}, t)$ to be invertible for all nonzero states and time, such that, by Theorem 2.2.1, $\mathcal{A}^{s\alpha}_{\mathbf{xf}} \neq \emptyset, \alpha = o, l, i$. Therefore, the applicabilities for both schemes are significantly widened by the proposed results.

Algorithm 4.6.1: The SDDRE scheme with feasible SDC selections and closed-form solutions, regarding the system (1.14) with performance index (1.15)

input : $\mathbf{x}, \mathbf{f}, B, C, R$ and t_f

output : $\mathbf{u}_{\text{SDDRE}}$

1 Construct an SDC Matrix in $\mathcal{A}^{si}_{\mathbf{xf}}$, either by Algorithm 2.3.1 to easily generate one or Theorems 3.2.1-3.2.2 for the other candidates.

2 Solve the corresponding SDDRE (1.16) with the explicit formula for P by Eq. (4.6).

3 $\mathbf{u}_{\text{SDDRE}} = -R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t)\mathbf{x}$ as in Eq. (1.17).

Algorithm 4.6.2: The SDRE scheme with feasible SDC selections and tunable performances, regarding the system (1.14) with performance index (1.18)

input : $\mathbf{x}, \mathbf{f}, B, C,$ and R

output : $\mathbf{u}_{\text{SDRE}}$

1 Construct an SDC Matrix in $\mathcal{A}^{s\alpha}_{\mathbf{xf}}, \alpha = o, l, i$, either by Algorithm 2.3.1 to easily generate one or Theorems 3.2.1-3.2.2 for the other candidates.

2 Solve the corresponding SDRE (1.16) by Algorithm 4.2.1.

3 $\mathbf{u}_{\text{SDRE}} = -R^{-1}(\mathbf{x}, t)B^T(\mathbf{x}, t)P(\mathbf{x}, t)\mathbf{x}$ as in Eq. (1.20).

Chapter 5

Illustrative Examples [1]

To further endorse the theoretical results in Chapters 2-4, various examples are given to demonstrate the effectiveness of the proposed scheme via the SDRE/SDDRE approach, from both the illustrative point of view (the planar cases) and real-world applications (higher-order systems). Specifically, we will demonstrate the proposed analytical way of selecting SDC matrices as in Algorithm 2.3.1 and Theorems 3.2.1-3.2.2, such that the solvability condition of the SDRE as in Theorem 2.2.1 is satisfied. Therefore the SDRE scheme could be continued, especially when the states'values are such that the already chosen SDCs fail to provide a solution.

Note that, all the demonstrations are simulated via MATLAB® 2013, and the built-in function "ode45" is used to solve the ordinary differential equations.

5.1 Significant Planar Cases

Example 5.1.1. Consider the following system [13] via the SDRE scheme

$$\dot{x}_1 = x_1 x_2 \quad \text{and} \quad \dot{x}_2 = -x_2 + u. \tag{5.1}$$

Clearly, this system is in the form of (1.14) with $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{f}(\mathbf{x}) = [x_1x_2, -x_2]^T$ and $B(\mathbf{x}) = [0, 1]^T$. System (5.1) is stabilizable and two global stabilizers, one using the Sontag formula with the control Lyapunov function

[1] Journal and conference versions at $[152, 153, 159, 160]$

$V(x_1, x_2) := (x_1^2 e^{2x_2} + x_2^2)/2$ [225] and the other adopting the BS scheme [131], have the following forms:

$$u_{\text{Sontag}} = \frac{x_2^2 - \sqrt{x_2^4 + (x_1^2 e^{2x_2} + x_2)^4}}{x_1^2 e^{2x_2} + x_2} \tag{5.2}$$

$$\text{and} \quad u_{\text{BS}} = (1-\psi)x_2 - (1+\psi)x_1^2 - 2x_1^2 x_2, \ \psi > 0. \tag{5.3}$$

To demonstrate the SDRE design, we choose $Q(\mathbf{x}) = I_2$, $R(\mathbf{x}) = 1$ and an intuitive SDC matrix $A(\mathbf{x})$ with $a_{11}(\mathbf{x}) = a_{21}(\mathbf{x}) = 0$, $a_{12}(\mathbf{x}) = x_1$ and $a_{22}(\mathbf{x}) = -1$. Obviously, $(A(\mathbf{x}), B(\mathbf{x}))$ is stabilizable everywhere, and except the X_2-axis the SDRE solvability condition is violated; however, because $C(\mathbf{x})\mathbf{x} = \mathbf{x} \neq \mathbf{0}$ and by Theorem 2.2.1, at every nonzero state $\mathcal{A}_{\mathbf{xf}}^{s\gamma} \neq \emptyset$ for $\gamma = o, d, i$ (i.e. observable, detectable, and having no unobservable mode on the imaginary axis). When $\mathbf{x} = [0, x_2]^T$ and $x_2 \neq 0$, $\mathbf{f} = [0, -x_2]^T = -\mathbf{x}$ and, by Theorems 3.2.1-3.2.2, $\mathcal{A}_{\mathbf{xf}}^{so} = \mathcal{A}_{\mathbf{xf}}^{sd} = \mathcal{A}_{\mathbf{xf}}^{si} = \mathcal{A}_{\mathbf{xf}}^{s} = \mathcal{A}_{\mathbf{xf}} \backslash \mathcal{A}_{\mathbf{xf}}^{\bar{s}} = \{A \mid a_{11} < 0,\ a_{12} = 0,\ a_{21} \in \mathbb{R} \ \&\ a_{22} = -1\}$. In the following, we will choose $a_{11} = -1$ and $a_{21} = 0$ for the SDC matrix of the SDRE scheme when $\mathbf{x} \in X_2$-axis.

Numerical results for initial states $\mathbf{x}(0) = [1, 1]^T$ are summarized in Fig. 5.1 and Table 5.1, where we have adopted the following three controllers: u_{Sontag} (labeled Sontag), u_{BS} with $\psi = 2$ (labeled BS) and the SDRE controller (labeled SDRE). It is observed from Fig. 5.1 that all of the system states of the three schemes converge to zero and, from Table 5.1, the SDRE scheme has better performances than the other two schemes in the performance indices that are listed in Table 5.1, where $||u||_\infty := \max_t ||u||$ denotes the maximum control magnitude that is required during the control period and the integration is evaluated from $t = 0$ to $t = 1000$.

It is noted that the solution trajectories of the three schemes remain on the X_2-axis if they start from there, because $\dot{x}_1 = x_1 x_2|_{x_1=0} = 0$. Thus, the trajectories of the three schemes will never reach the X_2-axis unless they start from there. By direct calculation, $u_{\text{Sontag}} = u_{\text{SDRE}} = (1-\sqrt{2})x_2$ and $u_{\text{BS}} = (1-\psi)x_2$ if the system state starts from the X_2-axis. The resulting closed-loop dynamics for x_2 are $\dot{x}_2 = -\psi x_2$ for the BS design and $\dot{x}_2 = -\sqrt{2}x_2$ for both the Sontag and SDRE schemes. It is interesting to note that, when $\mathbf{x} \in X_2$-axis, u_{SDRE} remains unchanged regardless of the choice of $A(\mathbf{x}) \in \mathcal{A}_{\mathbf{xf}}^{s}$; however, if the weighting matrices are changed to be $Q(\mathbf{x}) = \text{diag}(q_1, q_2) > 0$ and $R(\mathbf{x}) = r > 0$, then $u_{\text{SDRE}} = (1 - \sqrt{1 + q_2/r})x_2$ and the resulting closed-loop dynamics for x_2 becomes $\dot{x}_2 = -\sqrt{1 + q_2/r} \cdot x_2$, both are independent of q_1. Moreover, $u_{\text{SDRE}} \approx 0 = u_{\text{BS}}|_{\psi=1}$ when $r \gg q_2$, which implies that the control effort should be reduced as much as possible.

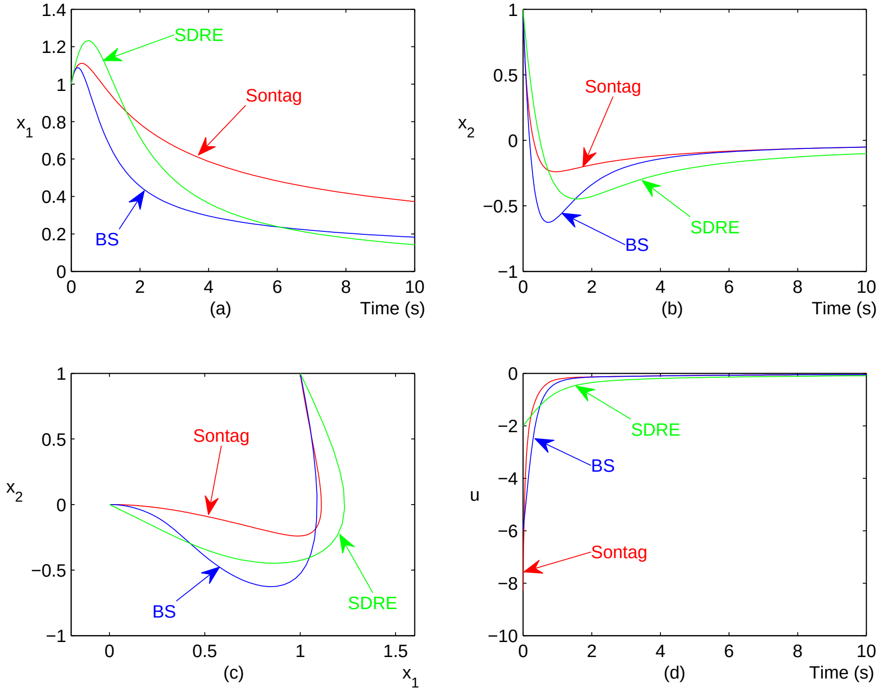

Figure 5.1: Example 5.1.1 - Time history of the system states and control inputs.

Table 5.1: Example 5.1.1 - Performances of the three schemes.

	Final time of $\mathbf{x}^T\mathbf{x} = 0.01$	$\int(\mathbf{x}^T\mathbf{x} + u^2)$	$\int u^2$	$\|u\|_\infty$
Sontag	3.2×10^3	13.6	3.4	8.3
BS	8.3×10^2	9.7	5.8	6
SDRE	86.3	6.1	2.2	2

Example 5.1.2. Revisit the system [13] as in Ex. 5.1.1, but via the SDDRE scheme

$$\dot{x}_1 = x_1 x_2 \text{ and } \dot{x}_2 = -x_2 + u. \tag{5.4}$$

Clearly, this system is in the form of (1.14) with $\mathbf{f}(\mathbf{x}) = [x_1 x_2, -x_2]^T$ and $B(\mathbf{x}) = [0, 1]^T$. Applying Algorithm 4.6.1 for the refined SDDRE scheme with $Q = 10^{-2} \cdot I_2$, $R = 10^{-1}$, $S = 10 \cdot I_2$, $t_0 = 0$ (s) and $t_f = 6$ (s), it can be seen in Fig. 5.2 that the origin is stabilizable under the SDDRE scheme for both initial states ($\mathbf{x}_0 = [-1, 1]^T, [-3, 3]^T$), which is simulated via MATLAB®.

Moreover, it is worth mentioning that the trajectories starting from $\mathbf{x}_0 = [-3, 3]^T$ (the blue line) seem to consume much more energy than from $\mathbf{x}_0 = [-1, 1]^T$ (the red line), as in (d) of Fig. 5.2. This is because $(x_1, 0), x_1 \in \mathbb{R}$ (the X_1 axis) are the equilibrium points with respect to the dynamics (5.4), and when the system trajectories cross the X_1-axis, it may take much effort to pull the system state to the origin, especially when the trajectories are farther away from the origin. Note that the same phenomena happens for the system trajectories starting from $\mathbf{x}_0 = [-1, 1]^T$ and those in Ex. 5.1.1, which cost less energy since all the crossing with the X_1-axis is much closer to the origin, as compared to the behavior starting from $\mathbf{x}_0 = [-3, 3]^T$.

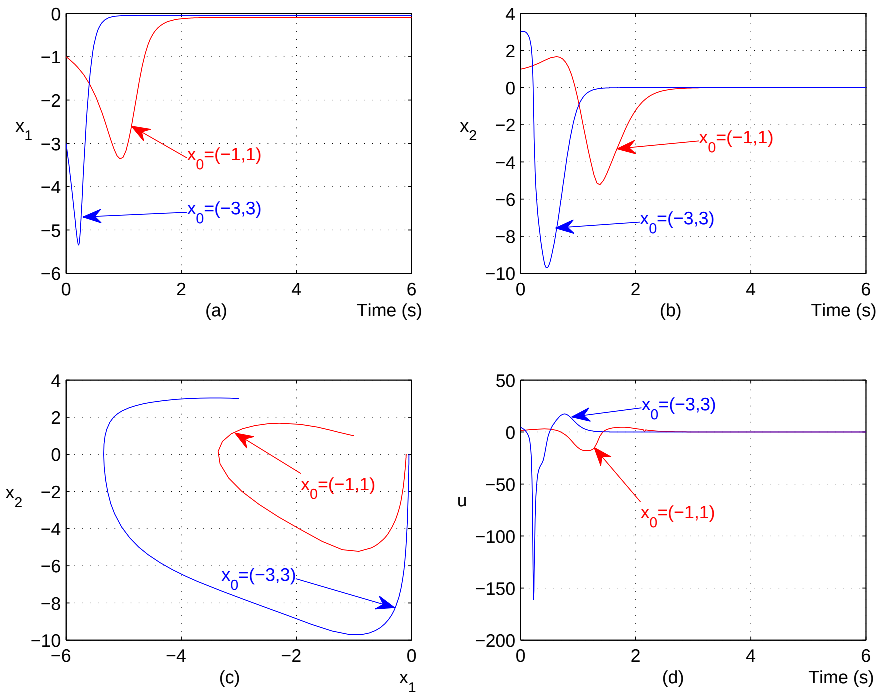

Figure 5.2: Example 5.1.2 - Time history of the system states and control inputs.

Example 5.1.3. Consider the following system [13]

$$\dot{x}_1 = -x_2 \quad \text{and} \quad \dot{x}_2 = x_1 + x_2 u. \tag{5.5}$$

Again, this system is in the form of (1.14) with $\mathbf{f}(\mathbf{x}) = [-x_2, x_1]^T$ and $B(\mathbf{x}) = [0, x_2]^T$. This system is stabilizable, and the Sontag stabilizer using the control Lyapunov function $V(x_1, x_2) := (x_1^2 + x_2^2)/2$ becomes

$$u_{Sontag} = -x_2^2. \tag{5.6}$$

With $Q(\mathbf{x}) = I_2$, $R(\mathbf{x}) = 1$ and the SDC matrix

$$A(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.7}$$

it is easy to check that $(A(\mathbf{x}), B(\mathbf{x}))$ is stabilizable and detectable everywhere except for the nonzero points of X_1-axis, where $(A(\mathbf{x}), B(\mathbf{x}))$ is unstabilizable. Thus, the SDRE scheme with the SDC matrix given by (5.7) will fail to operate if the system state reaches the X_1-axis. However, at every nonzero state $\mathbf{x} = [x_1, 0]^T$ of the X_1-axis, we have $\mathbf{f} = [0, x_1]^T$ and $\{\mathbf{x}, \mathbf{f}\}$ are linearly independent. Thus, by Theorem 2.2.1, Problem B is always solvable over the nonzero X_1-axis and a matrix A exists there that can both continue the SDRE scheme and render the closed-loop matrix Hurwitz. According to Algorithm 2.3.1, we may select the matrix A in the form of (2.2) with

$$\lambda_1 = -1, \ \lambda_2 = -2 \text{ and } \mathbf{q}_i^T = \text{null}(\lambda_i \mathbf{x} - \mathbf{f}), \ i = 1, 2 \tag{5.8}$$

whenever the system state reaches the X_1-axis.

Fig. 5.3 demonstrates the success of the schemes with the initial state being chosen to be $[5, 5]$. Once again, the required maximum control magnitude for the Sontag controller is $||u||_\infty = 25$, which is much larger than that of the SDRE design $||u||_\infty = 1.66$. Figs. 5.4-5.5 illustrates the issue of the computational performance as discussed in Section 4.2, with the red (resp. blue) line corresponding to the result by MATLAB® built-in solver (resp. the proposed alternative). Note that for the alternative method, the different thresholds (i.e. 10^{-7} and 10^{-14}) in Figs. 5.4-5.5 are the prescribed accuracy for stopping the method [95], i.e. if the computed solution results the residual of the corresponding SDRE being less than the threshold, then the algorithm stops. As seen from Figs. 5.4-5.5, both methods yield quite acceptable accuracy but differ significantly in the computation time. For the MATLAB® built-in

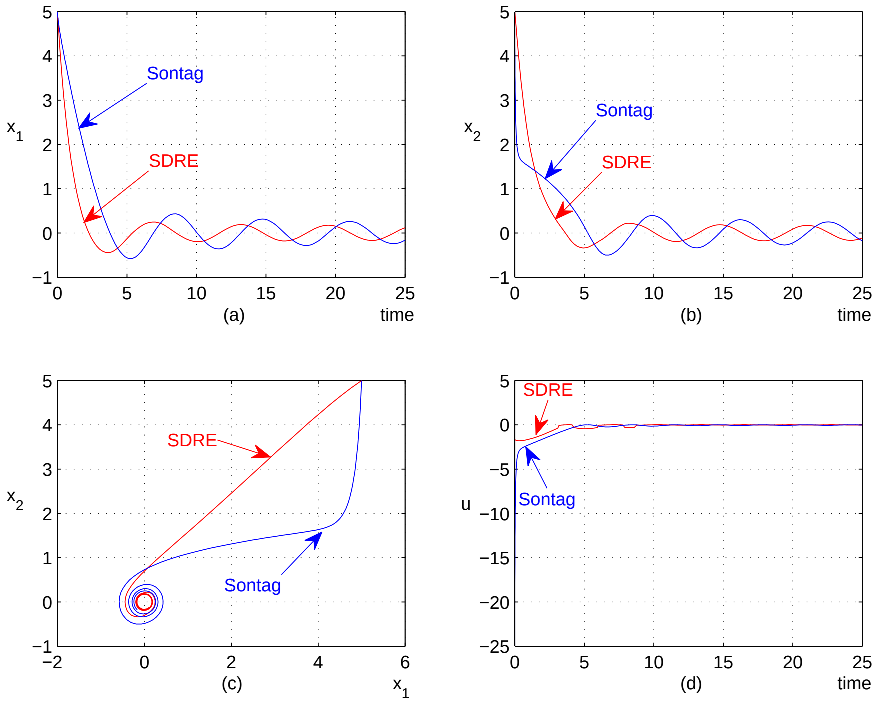

Figure 5.3: Example 5.1.3 - Time history of the system states and control inputs.

solver, the accumulated computation time is around 82.7 sec.; while that for the alternative method with threshold as 10^{-7} (resp. 10^{-14}) being 32.6 (resp. 36.7) sec. Therefore, in this simulation setup, the proposed alternative results in better performance for the criteria of accuracy and computation time.

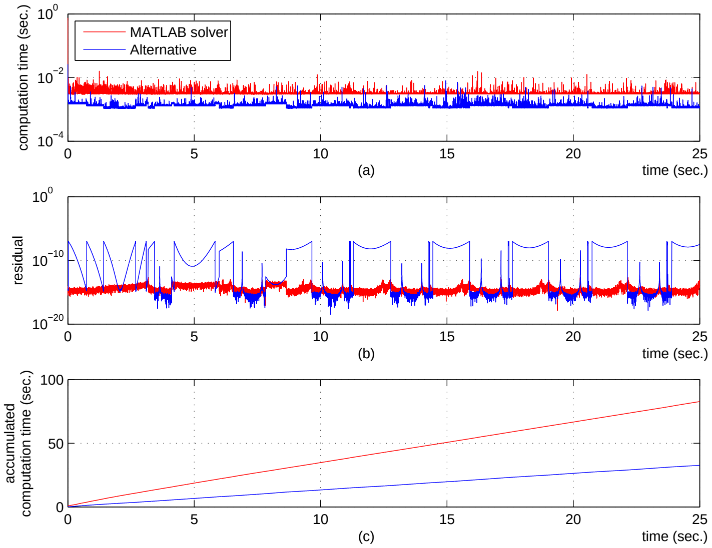

Figure 5.4: Example 5.1.3 - Comparison of the computational performance solving SDRE with threshold as 10^{-7}.

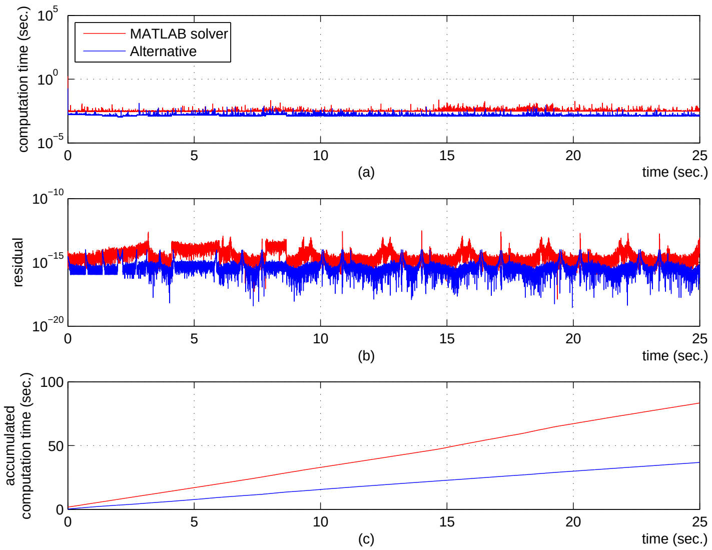

Figure 5.5: Example 5.1.3 - Comparison of the computational performance solving SDRE with threshold as 10^{-14}.

Example 5.1.4. Consider the following system [132]

$$\dot{x}_1 = x_1^2 x_2 - x_1 \ \text{ and } \ \dot{x}_2 = u. \tag{5.9}$$

Clearly, this system is in the form of (1.14) with $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{f}(\mathbf{x}) = [x_1^2 x_2 - x_1, 0]^T$ and $B(\mathbf{x}) = [0, 1]^T$. In [132], it is shown that with the control via the the feedback linearization (FL) method,

$$u_{FL} = -\gamma x_2, \ \text{where } \gamma > 0, \tag{5.10}$$

the domain of attraction is exactly the set $\{x_1 x_2 < 1 + \gamma\}$.

To demonstrate the SDRE design, we simply choose $Q(\mathbf{x}) = I_2$, $R(\mathbf{x}) = 1$ and an intuitive SDC matrix $A(\mathbf{x})$ with $a_{11}(\mathbf{x}) = x_1 x_2 - 1$ and the other entries being zero. Obviously, $(A(\mathbf{x}), B(\mathbf{x}))$ is stabilizable everywhere, except in the region $\{x_1 x_2 \geq 1\}$ where the SDRE solvability condition is violated; however, because $C(\mathbf{x})\mathbf{x} = \mathbf{x} \neq \mathbf{0}$ and by Theorem 2.2.1, at every nonzero state $\mathcal{A}_{\mathbf{xf}}^{s\gamma} \neq \emptyset$ for $\gamma = o, d, i$ (i.e. observable, detectable, and having no unobservable mode on the imaginary axis). From Algorithm 2.3.1, we may easily construct a feasible SDC matrix in the region $\{x_1 x_2 \geq 1\}$ as:

- in the region $\{x_1 x_2 > 1\}$, where $\{\mathbf{x}, \mathbf{f}\}$ are linearly independent

$$\lambda_1 = -1, \ \lambda_2 = -2, \ \text{and choose } \mathbf{q}_i \text{ such that}$$

$$\mathbf{q}_i^T (\lambda_i \mathbf{x} - \mathbf{f}) = 0, \forall i = 1, 2.$$

- on the hyperbola $\{x_1 x_2 = 1\}$, where $\{\mathbf{x}, \mathbf{f}\}$ are linearly dependent

$$\lambda_1 = -1, \ \lambda_2 = 0, \ \mathbf{q}_1 = [x_2, x_1]^T, \ \text{and } \mathbf{q}_2 = B + \frac{\mathbf{x}}{||\mathbf{x}||}.$$

Note that in the region $\{x_1 x_2 < 1\}$, we just adopt the above-mentioned intuitive SDC matrix. The effectiveness of such a multi-SDC SDRE scheme is illustrated in Fig. 5.6, where all trajectories converge to the origin. For clarity, in Fig. 5.6 we mark the initial state residing in the region $\{x_1 x_2 \geq 1\}$ with a red circle; while in the other region with a blue star.

The performance comparison with initial state (0.9,1.2) of the controls by SDRE and FL are summarized in Fig. 5.7 and Table 5.2, labeled SDRE and FL, respectively. For FL, we choose $\gamma = 1.5$ such that the overall trajectories are included in the DOA. It is observed from Fig. 5.7 that both the system states of the two schemes converge to the origin. In Table 5.2, we consider the following criteria: (1) the convergence time, defined as the final time when $\mathbf{x}^T\mathbf{x} < 10^{-3}$; (2) the performance index $\int(\mathbf{x}^T\mathbf{x} + u^2)$; (3) the energy consumption $\int u^2$; and (4) the maximum control magnitude $||u||_\infty := \max_t |u|$, with (2)-(4) being evaluated from the beginning to the time when $\mathbf{x}^T\mathbf{x} < 10^{-3}$. For the considered criteria, the FL scheme seems to converge slightly faster than the SDRE scheme, but at the expenses of the control efforts and performance as defined in criteria (2)-(4).

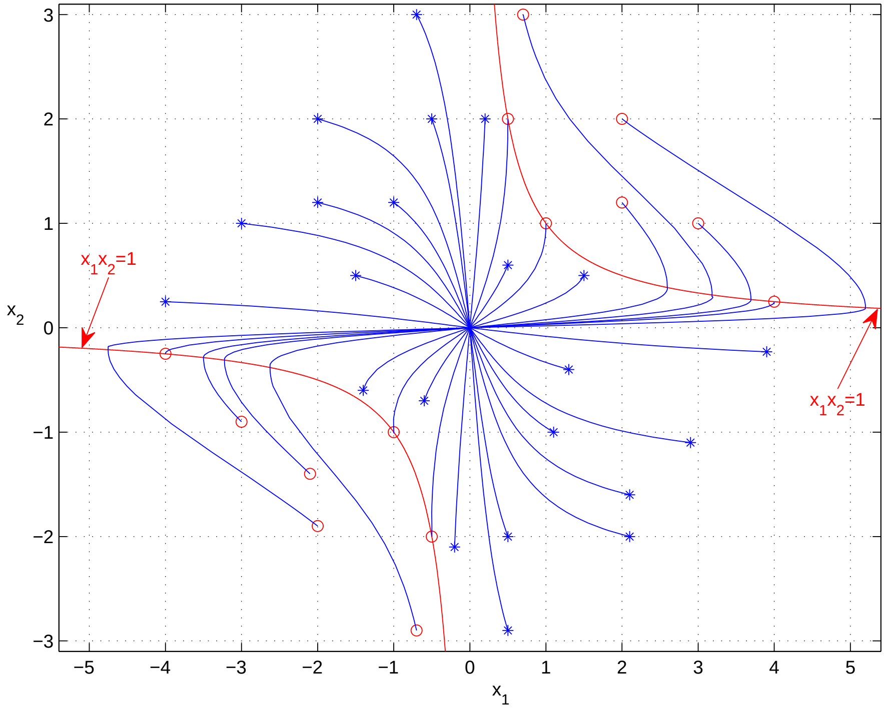

Figure 5.6: Example 5.1.4 - Phase plot of various initial states.

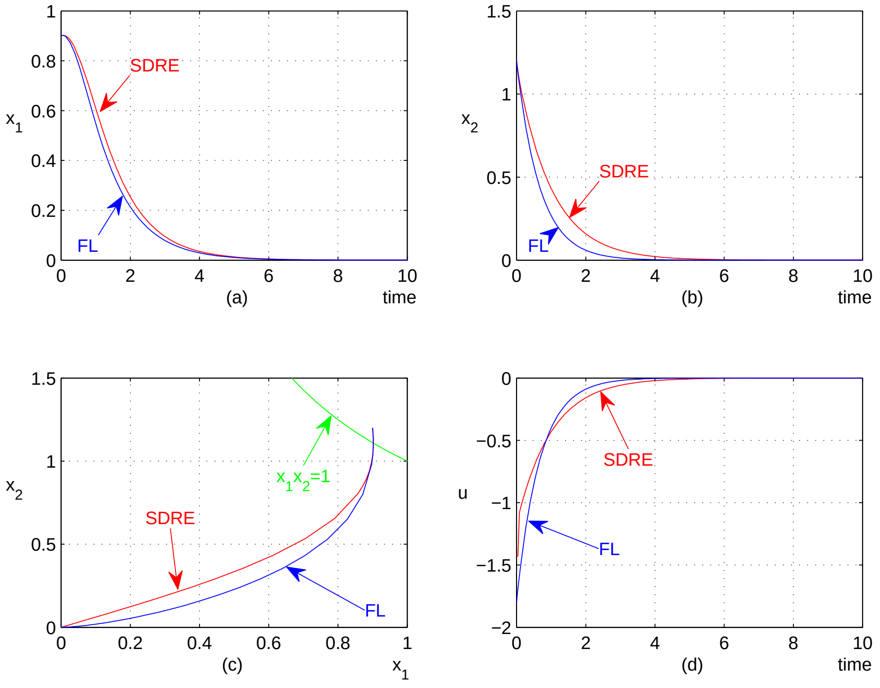

Figure 5.7: Example 5.1.4 - Time history of the system states and control inputs.

Table 5.2: Example 5.1.4 - Performances of the two schemes.

	Final time when $\mathbf{x}^T\mathbf{x} = 10^{-3}$	$\int(\mathbf{x}^T\mathbf{x} + u^2)$	$\int u^2$	$\|u\|_\infty$
SDRE	7.88	2.039	0.642	1.434
FL	7.529	2.047	0.925	1.8

5.2 Real-World Applications

5.2.1 Reliable Satellite Attitude Stabilization

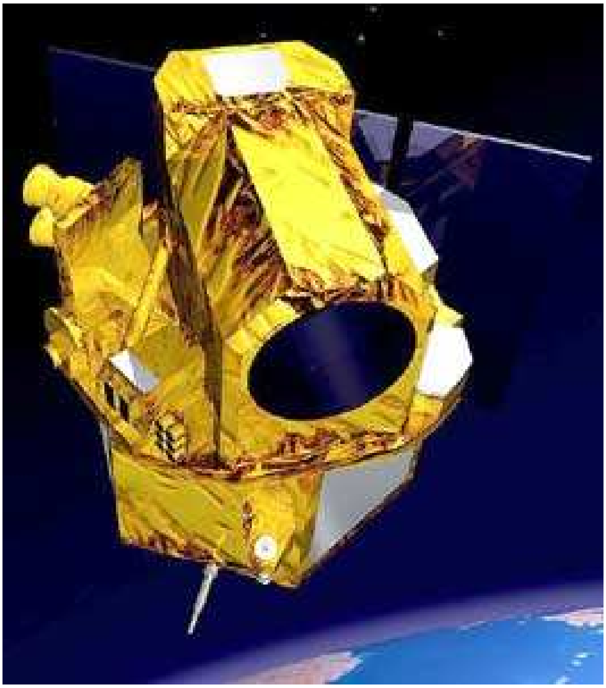

Figure 5.8: Example 5.2.1 - The Earth observation satellite ROCSAT 2 [2, 4].

Consider an attitude model for a spacecraft along a circular orbit as in Fig. 5.8 [4, 46, 154, 156, 161, 186]. The three Euler's angles (ϕ, θ, ψ) and their derivatives are adopted as the six state variables. To demonstrate the proposed scheme, we assume that the thruster is the only applied control force. Note that in the original 2nd-order nonlinear dynamics, there are four actuators available and any three of them would make the system to be controllable *pointwisely*. However in this simulation, we assume only two actuators are available/healthy to demonstrate the reliability of the SDRE scheme. Given $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6)^T = (\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi})^T$, $\mathbf{u} = (u_1, u_2)^T$,

$\mathbf{f}(\mathbf{x}) = (x_4, x_5, x_6, f_4(\mathbf{x}), f_5(\mathbf{x}), f_6(\mathbf{x}))^T$, and $B = [0_{2\times 3},\ G^T]^T$, where

$$\begin{aligned}
f_4(\mathbf{x}) \;=\;& \omega_0 x_6 cx_3 cx_2 - \omega_0 x_5 sx_3 sx_2 + \frac{I_y - I_z}{I_x}\Big[x_5 x_6 + \omega_0 x_5 cx_3 sx_1 \\
& + \omega_0 x_5 cx_1 sx_3 sx_2 + \omega_0 x_6 cx_3 cx_1 + \frac{1}{2}\omega_0^2 s(2x_3) c^2 x_1 sx_2 \\
& + \frac{1}{2} w_0^2 c^2 x_3 s(2x_1) - \omega_0 x_6 sx_3 sx_2 sx_1 - \frac{1}{2}\omega_0^2 s^2 x_2 s^2 x_3 s(2x_1) \\
& - \frac{1}{2}\omega_0^2 s(2x_3) sx_2 s^2 x_1 - \frac{3}{2}\omega_0^2 c^2 x_2 s(2x_1)\Big], \qquad (5.11)
\end{aligned}$$

$$\begin{aligned}
f_5(\mathbf{x}) \;=\;& \omega_0 x_6 sx_3 cx_1 + \omega_0 x_4 cx_3 sx_1 + \omega_0 x_6 cx_3 sx_2 sx_1 \\
& + \omega_0 x_5 sx_3 cx_2 sx_1 + \omega_0 x_4 sx_3 sx_2 cx_1 + \frac{I_z - I_x}{I_y} \\
& \cdot \Big[x_4 x_6 + \omega_0 x_4 cx_1 sx_3 sx_2 + \omega_0 x_4 cx_3 sx_1 - \omega_0 x_6 sx_3 cx_2 \\
& - \frac{1}{2}\omega_0^2 s(2x_2) s^2 x_3 cx_1 - \frac{1}{2} w_0^2 cx_2 sx_1 s(2x_3) \\
& + \frac{3}{2}\omega_0^2 s(2x_2) cx_1\Big], \qquad (5.12)
\end{aligned}$$

$$\begin{aligned}
f_6(\mathbf{x}) \;=\;& \omega_0 x_4 sx_1 sx_3 sx_2 - \omega_0 x_6 cx_1 cx_3 sx_2 - \omega_0 x_5 cx_1 sx_3 cx_2 \\
& + \omega_0 x_6 sx_3 sx_1 - \omega_0 x_4 cx_3 cx_1 + \frac{I_x - I_y}{I_z}\Big[x_4 x_5 + \omega_0 x_4 cx_3 cx_1 \\
& - \omega_0 x_4 sx_3 sx_2 sx_1 - \omega_0 x_5 sx_3 cx_2 - \frac{1}{2}\omega_0^2 s(2x_3) cx_2 cx_1 \\
& + \frac{1}{2} w_0^2 s^2 x_3 sx_1 s(2x_2) - \frac{3}{2}\omega_0^2 s(2x_2) sx_1\Big], \qquad (5.13)
\end{aligned}$$

$$\text{and } G = \begin{pmatrix} 0.67 & 0.69 & 0.28 \\ 0.67 & -0.69 & 0.28 \end{pmatrix}^T. \qquad (5.14)$$

Here, I_x, I_y, and I_z are the inertias with respect to the three body coordinate axes, ω_0 denotes the constant orbital rate, and c (resp., s) denotes the cosine (resp., sine) function. In this study, we assume that $I_x = I_z = 2000\ \mathrm{kg}\cdot\mathrm{m}^2$,

$I_y = 400$ kg $\cdot$ m^2 and $\omega_0 = 1.0312 \times 10^{-3}$ rad/s, and the angular positions are constrained to be $x_1, x_3 \in [-\pi, \pi]$ and $x_2 \in [-\pi/2, \pi/2]$.

For the SDRE scheme, we choose $Q(\mathbf{x}) = diag(10^2, 10^2, 10^2, 10^2, 1, 10)$ and $R(\mathbf{x}) = 0.1 \cdot I_2$. To illustrate the proposed scheme, at first we follow the general guidelines summarized in [51] to construct an SDC matrix, such as the following factorization: (Appendix A.2 includes the specific representation)

$$
\begin{aligned}
& 2\cos^2(x_3)\sin(2x_1) \\
= \; & \cos^2(x_3)\frac{\sin(2x_1)}{x_1}x_1 + \frac{\cos^2(x_3)-1}{x_3}\sin(2x_1)x_3 + \frac{\sin(2x_1)}{x_1}x_1 \\
= \; & \begin{bmatrix} \frac{\sin(2x_1)}{x_1} + \cos^2(x_3)\frac{\sin(2x_1)}{x_1} & 0 & \frac{\cos^2(x_3)-1}{x_3}\sin(2x_1) & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.
\end{aligned}
$$

Note that every state component in the original dynamics contributes as an corresponding element in $A(\mathbf{x}, t)$, i.e. capture their state dependency in the proper entry of SDC matrix. When the SDRE solvability condition of such SDC matrix is violated (about 21 violations in Fig. 5.9), which could be easily checked via the combination of Sections 2.2 and 3.1 (solution to Problem 1), we resort to the proposed scheme to construct alternatively a feasible SDC matrix, either by Algorithm 2.3.1 to easily generate one or Theorems 3.2.1-3.2.2 for the other candidates. Note that the existence of such feasible SDC matrix is guaranteed for all nonzero states by Theorem 2.2.1, since $Q(\mathbf{x})$ is chosen to be nonsingular and thus $C(\mathbf{x}) \cdot \mathbf{x} \neq 0$, $\forall \mathbf{x} \in \mathbb{R}^6 \backslash \{0\}$. It is observed from Fig. 5.9 that such multi-SDC SDRE scheme stabilizes the spacecraft's attitude, with only two healthy actuators. The red lines correspond to the time instants when feasible SDC matrices are constructed to continue the SDRE scheme successfully.

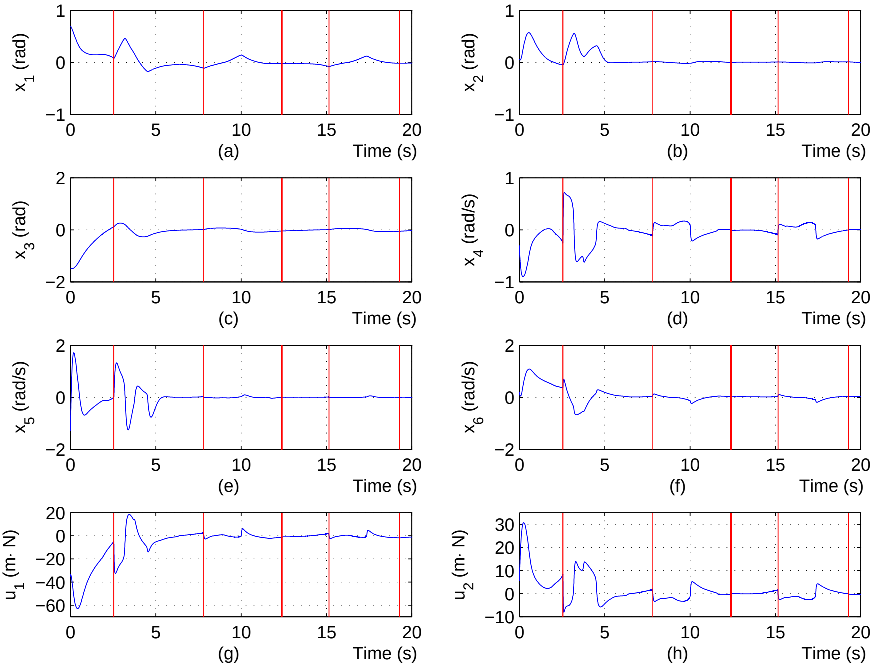

Figure 5.9: Example 5.2.1 - Time history of the six system states and the two control inputs.

5.2.2 Robust Vector Thrust Control Using The Caltech Ducted Fan

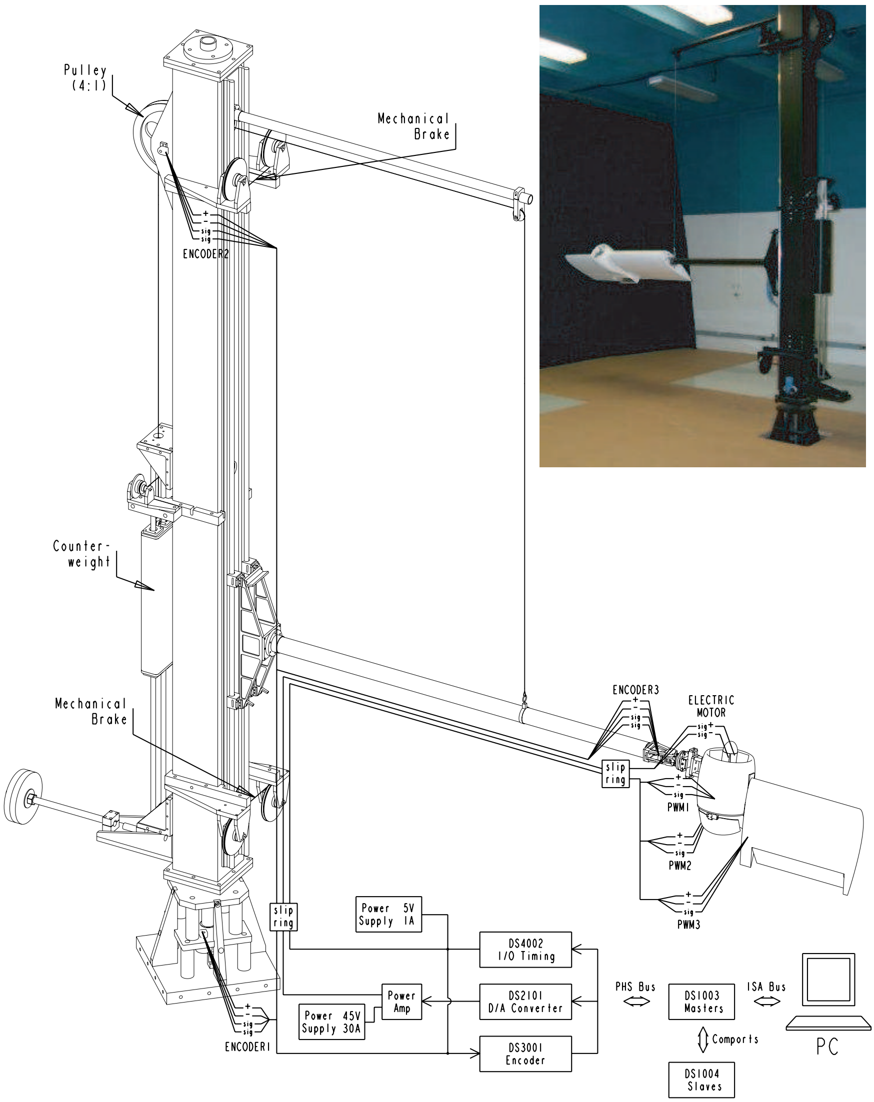

Figure 5.10: Example 5.2.2 - Caltech ducted fan testbed [170].

In this section, the Caltech Ducted Fan in Fig. 5.10 is adopted for demonstrating the effectiveness of the proposed scheme. The dynamics of this small flight control equipment are representative of various aerial applications equipped with the cutting-edge vector thrust control, such as the aircrafts F18-HARV (United States, manufactured by McDonnell Douglas) and Rockwell-MBB X-31 (United States and Germany, manufactured by Rockwell/Messerschmitt-Bölkow-Blohm), and the missiles MICA (France, manufactured by MBDA) and MIM-104F (PAC-3, United States, manufactured by Raytheon, Hughes and RCA). Due to significant outperformances than the traditional fixed-angle thrust, recently numerous defence-related authorities around the world are devoting efforts in developing this technology.

A diagram of the experimental systems is shown in Fig. 5.11, which consists of a ducted fan engine with a electric motor and 6-inch diameter blade. Flaps of the fan allow the thrust to be vectored from side to side and even reversed [181]. Such testbed originated from scholars and researchers in Caltech, with a number of related contributions as follows. [47] describes the overall design and control considerations. [130] performed and compared several different linear and nonlinear controllers, with a more focused comparison on the LPV method. [241,242] used the differential flatness-based controllers. To sum up their works [21,25,26,80,171,172], R. M. Murray summarized a brief note [181], including the description of the dynamics and the corresponding parameter values. Note that the detailed derivations of the dynamics could be found in [170].

A simplified planar model in Fig. 5.12 is widely adopted for demonstration (e.g. [123, 255]), which model ignores the stand dynamics but is useful for determining basic characteristics and testing initial designs. In the following demonstration, a much more accurate description of the dynamics [181] is considered, which includes a simplified stand dynamics with most of the important effects and is described as below. Denote the state variables as $[x_1, x_2, x_3, x_4, x_5, x_6] = [x, \dot{x}, y, \dot{y}, \theta, \dot{\theta}]$, where (x, y, θ) are the position and orientation of a point on the main axis of the fan that is distance l from the center of mass as in Fig. 5.13. Besides, the control forces acting on the fan are assumed to be (u_1, u_2) with u_1 perpendicular to the axis of the fan acting at a distance r, while u_2 parallel to the axis of the fan. Moreover, denote $\phi_2 = \frac{x_3}{r_f}$, and the dynamics of equation is

$$
\begin{aligned}
B_{21} &= \frac{\cos(\phi_2)\cos(x_5)}{\alpha - \delta\sin^2(\phi_2)},\ B_{22} = \frac{-\cos(\phi_2)\sin(x_5)}{\alpha - \delta\sin^2(\phi_2)}, \\
B_{41} &= \frac{\cos(\phi_2)\sin(x_5)}{\beta},\ B_{42} = \frac{\cos(\phi_2)\cos(x_5)}{\beta}, \\
B_{61} &= \frac{r}{J},\ \text{and the rest elements are zero,}
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
f_1 &= x_2,\ f_3 = x_4,\ f_5 = x_6, \\
f_2 &= \left[2\left(\frac{\delta x_2 x_4}{r_f}\right)\sin(\phi_2)\cos(\phi_2) - \left(\frac{J_m \Omega x_2}{r_f}\right)\sin(x_5)\right] \cdot \\
&\quad \frac{1}{\alpha - \delta\sin^2(\phi_2)}, \\
f_4 &= -\left(\frac{\delta x_2^2}{r_f \beta}\right)\sin(\phi_2)\cos(\phi_2) - \gamma\cos(\phi_2) - \kappa\sin(\phi_2), \\
f_6 &= \left(\frac{J_m \Omega x_2}{J r_f} - \frac{m_f g l}{J}\right)\sin(x_5),
\end{aligned}
\tag{5.16}
$$

where the descriptions and values of all the parameters could be found in [181]. Note that, furthermore we include the dummy variable x_7 with the following dynamics

$$
\dot{x}_7 = -\lambda x_7,\ 0 < \lambda << 1, \tag{5.17}
$$

to account for the presence of the state-independent term (i.e. $\gamma\cos(\phi_2)$ in f_4 being nonzero at the origin). This conforming technique is adopted from [49,51]. Note that this conforming step (and thus the dummy variable x_7) is necessary for the fixed SDC factorization, but not for the proposed scheme.

The control objective is to drive both the state variables (x_1, x_3) to the origin [255]. For the simulation setup, we choose the significantly more challenging initial condition as $\left(5, 5, 5, 0, \frac{-0.9\pi}{2}, 0\right)$ [255], and $R = 0.1 \cdot I_2$, $Q = \mathrm{diag}\left(\left[1, 10^{-2}, 1, 10^{-2}, 10^{-2}, 10^{-2}\right]\right)$ with additional diagonal element 10^{-4} for the dummy variable x_7, and all the parameter values could be found from Table 2 in [181]. To demonstrate the effectiveness of the proposed scheme, the fixed SDC matrix is adopted first and, if the solvability of the corresponding SDRE (1.19) is violated, then we resort to the proposed scheme for a feasible SDC matrix (either by Algorithm 2.3.1 to easily generate one or Theorems 3.2.1-3.2.2 for the other candidates) such that the SDRE (1.19) is solvable. Regarding the fixed SDC matrix, we adopt the two extreme cases: one containing the most trivial zeros and the other containing the fewest trivial zeros, with specific representations in Appendix A.3.1 and A.3.2, respectively.

Fig. 5.14 depicts two simulation scenarios using MATLAB R2013®, one for the nominal system (labeled SDRE), and the other combined with the ISMC strategy [38, 39] to completely nullify the effect of any possible disturbance/uncertainty in the thrust force (labeled SDRE+ISMC). The basic idea for the SDRE-ISMC scheme is that the system trajectory is completely controlled by the SDRE scheme, which is designed for the nominal system, while the effect of the considered disturbance/uncertainty is completely nullified by the extra control efforts added independently to the SDRE control law, which is designed for the disturbed system via the ISMC scheme [150,154,158]. For a demonstration, we assume that the disturbance is $0.1 \cdot (\sin(t), \cos(t))$, and the control law of the SDRE-ISMC scheme is

$$\mathbf{u} = \begin{cases} \mathbf{u}_{SDRE} & \text{if } \mathbf{s} = \mathbf{0} \\ \mathbf{u}_{SDRE} - \rho \cdot \frac{B^T B\mathbf{s}}{||B^T B\mathbf{s}||} & \text{if } \mathbf{s} \neq \mathbf{0} \text{ and } ||B^T B\mathbf{s}|| \geq \epsilon \\ \mathbf{u}_{SDRE} - \rho \cdot \frac{B^T B\mathbf{s}}{\epsilon} & \text{if } \mathbf{s} \neq \mathbf{0} \text{ and } ||B^T B\mathbf{s}|| < \epsilon \end{cases} \tag{5.18}$$

where $\epsilon = 10^{-2}$, $\rho = 10 \cdot \max_t ||\mathbf{d}||$ (if exists), and $\mathbf{s}$ is the sliding variable on the following sliding surface corresponding to $\mathbf{s} = \mathbf{0}$

$$\mathbf{s} = B^T \left(\mathbf{x}(t) - \mathbf{x}(t_0) - \int_{t_0}^{t} \{\mathbf{f}(\mathbf{x}(\tau)) + B(\mathbf{x}(\tau))\mathbf{u}_{SDRE}(\tau)\} \, d\tau \right). \tag{5.19}$$

Note that the control law (5.18) is slightly modified from [38] to alleviate the chattering effect.

From Fig. 5.14, it can be seen that the SDRE scheme successfully stabilizes the objective state variables (x_1, x_3), while the others are moving within a reasonable and acceptable range. It should be emphasized that, for both fixed SDC matrices in Appendix A.3.1 and A.3.2, the corresponding SDREs (1.19) are unsolvable at every time instant, and thus the proposed scheme is activated for a feasible SDC matrix throughout the whole considered time horizon. Note that the existence of such a feasible SDC matrix is guaranteed for all nonzero states by Theorem 2.2.1, since $Q(\mathbf{x})$ is chosen to be nonsingular and thus $C(\mathbf{x}) \cdot \mathbf{x} \neq 0$, $\forall \mathbf{x} \in \mathbb{R}^6 \backslash \{0\}$. In addition, the trajectories of both scenarios coincide and, from Fig. 5.15, the sliding variables for the ISMC scheme seem to be zero in the considered time horizon, both of which agree with the theoretical results [150, 154, 158].

As a comparison, the trajectories based on the simplified planar model via the SDRE scheme with a fixed SDC matrix are recalled in Fig. 5.16 [255], which seem to take more time to stabilize the objective state variables (x_1, x_3) and exhibit larger overshoot/undershoot than the proposed scheme. Besides, the phase portrait of (x_1, x_3) is plotted in Fig. 5.17, together with the result based on the simplified dynamics and via the fixed-SDC SDRE scheme [255] (the blue line), which seems to traverse a much larger region than the proposed scheme (the red line).

To conclude, the capability of an alternative SDC construction over the currently existing guidelines is demonstrated, which widens the applicability of the SDRE scheme to a certain level. Through the simulation setup, a vector thrust control on the Caltech ducted fan via the SDRE scheme is developed, with the combination of the ISMC design to acquire robustness. As a future research, the performance of the real-time application via the proposed scheme is considered to be very valuable.

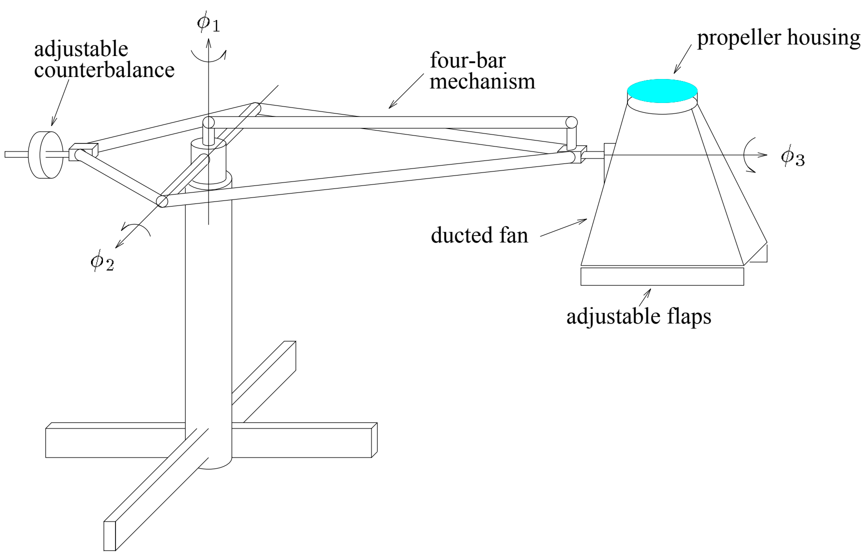

Figure 5.11: Example 5.2.2 - Caltech ducted fan with support stand [181].

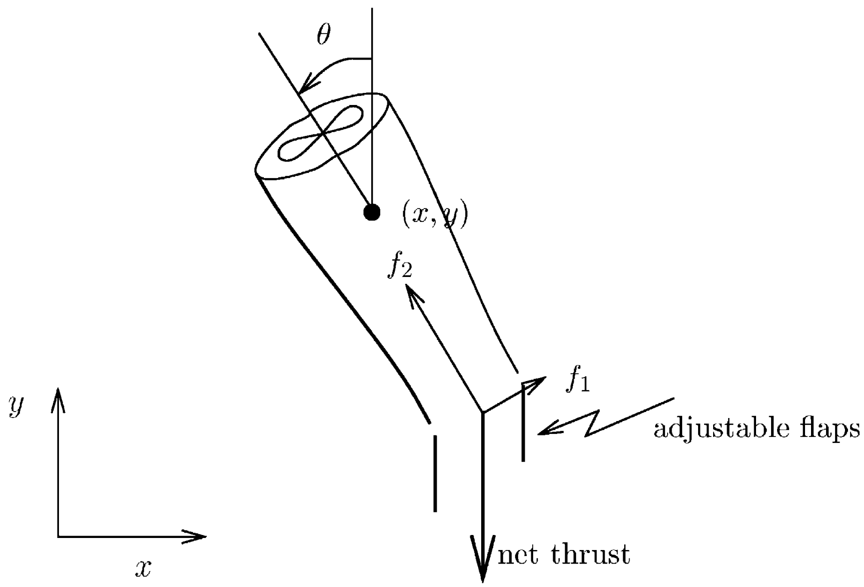

Figure 5.12: Example 5.2.2 - Planar ducted fan model [255].

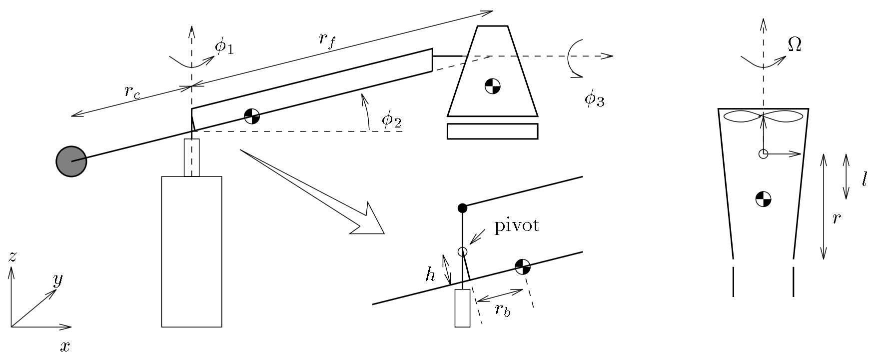

Figure 5.13: Example 5.2.2 - Ducted fan with simplified model of stand [181].

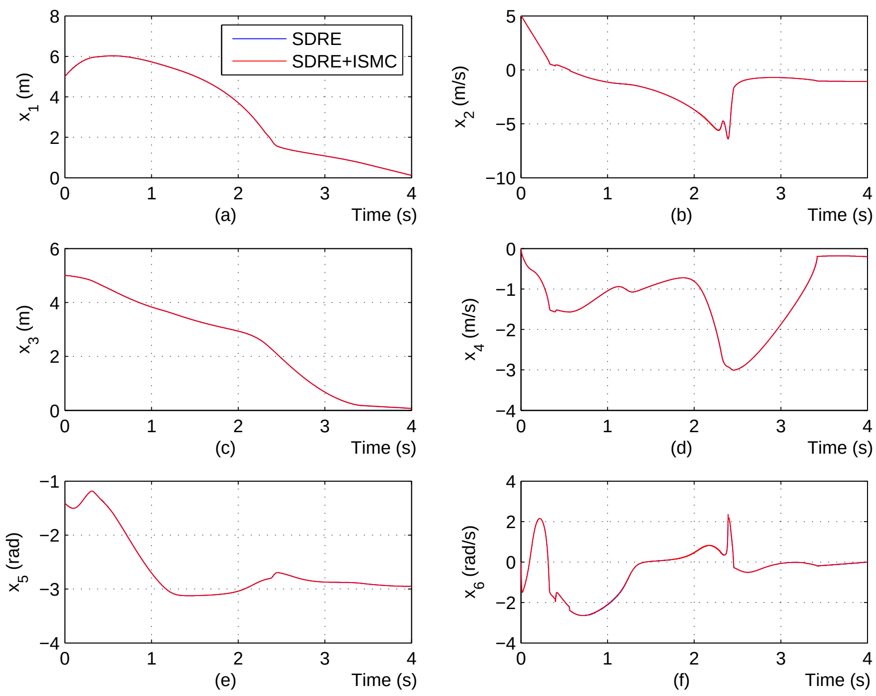

Figure 5.14: Example 5.2.2 - Time history of the six system states (the blue lines coincide with the red lines and are hidden behind).

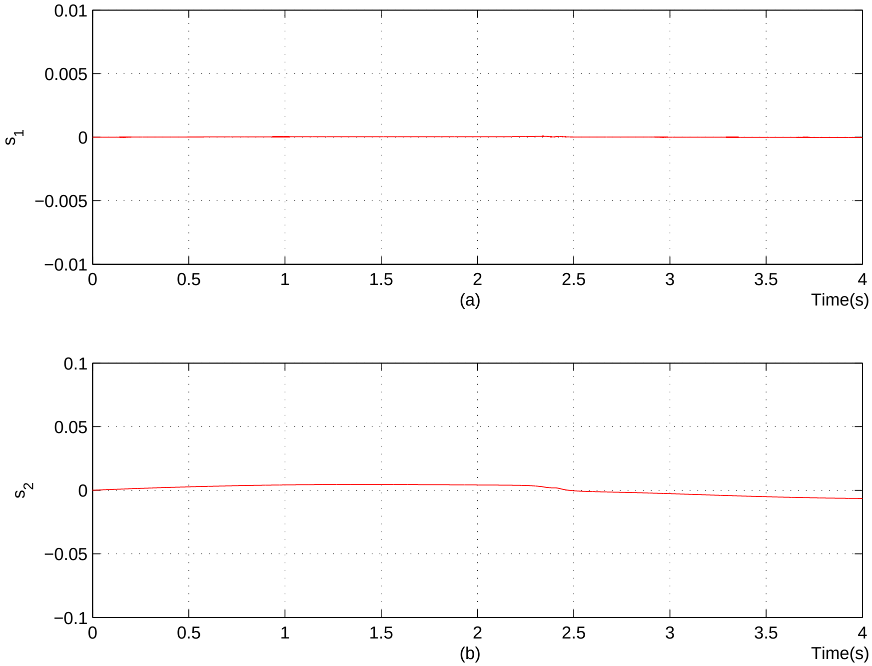

Figure 5.15: Example 5.2.2 - Time history of the sliding variables.

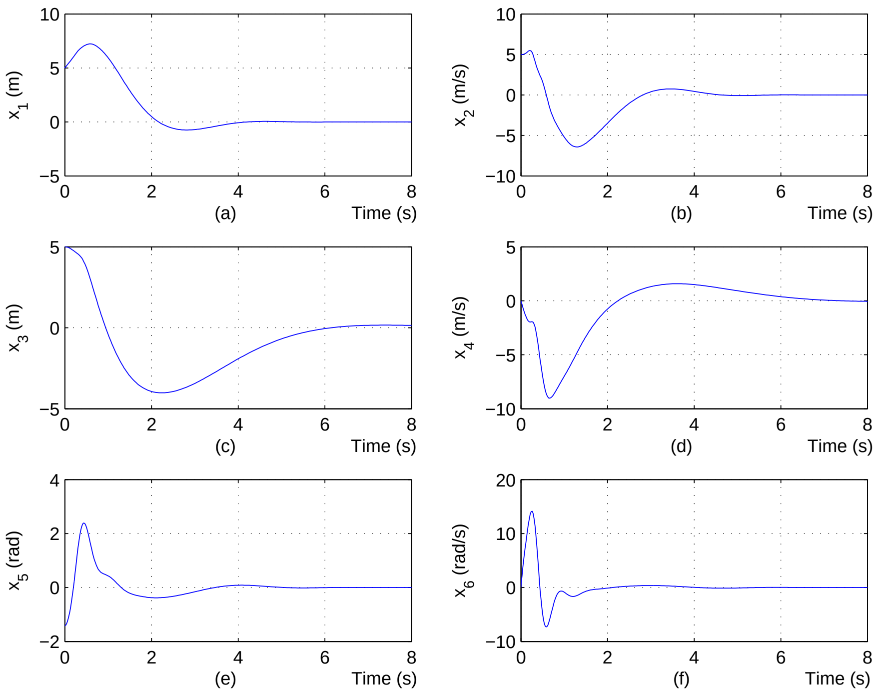

Figure 5.16: Example 5.2.2 - Time history of the six system states based on the simplified planar model [255].

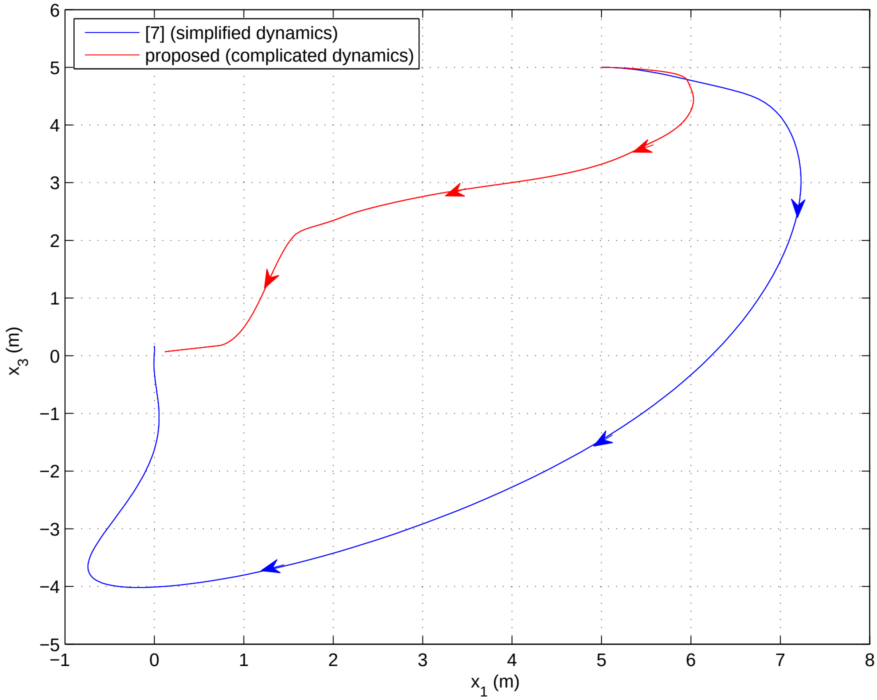

Figure 5.17: Example 5.2.2 - Trajectories in the state space x_1, x_3 for the proposed scheme (resp. [255]) with complicated (resp. simplified) dynamics.

5.2.3 Tracking Control of an Overhead Crane

Figure 5.18: Example 5.2.3 - Overhead Crane [3].

This section applies the proposed SDRE scheme on the tracking control of an overhead crane, using the integral servomechanism as described in Section 4.3. The overhead crane, as in Fig. 5.18, is a quite popular system for transporting heavy goods/equipments, commonly seen by a harbor or in a factory. The main function of the crane is to move objects from a specified place to another, therefore the tracking control behind the overall system seems to be the most crucial part.

A schematic of the experimental overhead crane is given in Fig. 5.19, which originates from the experimental testbed by scholar in KU Leuven (e.g. [114, 244]). In the following simulation, the tracking control by the SDRE scheme will be applied on such a testbed, acting as a demonstration of results in Section 4.3. The cart is actuated by a Servotube 1108 linear motor (Copley Controls) [244], which has an integrated incremental encoder measuring the position of the cart (x_C), with a resolution of 5 μm. The pendulum consists of a cylindrical load (mass $m = 1.3$ kg) hanging on two parallel cables, with one end of each cable mounted to a fixed point on the cart, while the other end connected to a winch mechanism. The winch mechanism consists of a pulley, and a

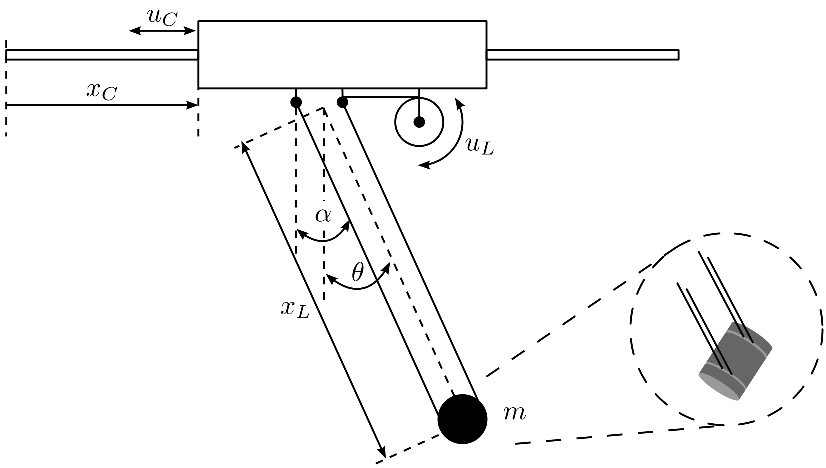

Figure 5.19: Example 5.2.3 - A diagram of the experimental overhead crane [244].

coupled DC motor (A-max 32 from Maxon Motors) with a gearbox reduction ratio 18. An incremental encoder (resolution: 500 pulses/rev.) is attached to the winch motor, yielding a resolution of the cable length measurement x_L of 2.15 μm. The maximum (resp. minimum) cable length is 0.95m (resp. 0.5m, for safety reasons). The angular deflection α of the left cable is measured with rotary incremental encoder BFH 19.05A40000-B2-5 (resolution: 40000 pulses/rev. from Baumer Electric). For a detailed description of the other variables/parameters, e.g. the angular deflection of the pendulum θ, could be found in [114, 244].

The dynamic model of the experimental overhead crane is given by [91,114,244], and is summarized as follows. Note that in [244], due to the fast dynamics of the velocity controller (as compared to the overall system dynamics), from the velocity controller setpoint u_C to the cart position x_C could be modeled as a pure integrator. However, a more accurate and complicated model (from recent results e.g. [114]) is available and thus adopted for the following demonstration, which consists of identified first-order models regarding the input-output relations of both the cart mechanism (u_C to x_C) and the winching mechanism (u_L to x_L). Denote the state variable $\mathbf{x} = (x_1, \cdots, x_8) = (x_C, v_C, x_L, v_L, \theta, \omega, u_C, u_L) \in \mathbb{R}^8$, $\mathbf{u} = (u_{CR}, u_{LR}) \in \mathbb{R}^2$, and the equation of motion is

$$
\begin{aligned}
f_1 &= x_2,\ f_3 = x_4,\ f_5 = x_6,\ f_7 = f_8 = 0,\\
f_2 &= \frac{1}{\tau_C}(A_C x_7 - x_2),\\
f_4 &= \frac{1}{\tau_L}(A_L x_8 - x_4),\\
f_6 &= -\frac{1}{x_3}(g \sin x_5 + 2x_4 x_6), \qquad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
B_{61} &= -\frac{A_C \cos x_5}{x_3},\\
B_{71} &= B_{82} = 1,\ \text{and the rest elements are zero,} \qquad (5.21)
\end{aligned}
$$

where the detailed description and values of all parameters could be found in Fig. 5.19 and [91, 114, 244]. To apply the integral servomechanism for tracking via SDRE as described in Section 4.3, let $\tilde{\mathbf{x}} = (\tilde{x}_1, \cdots, \tilde{x}_{10}) = (\mathbf{x}_I, \mathbf{x}_R, v_c, x_4, x_5, \cdots, x_8) \in \mathbb{R}^{10}$, where $\mathbf{x}_R = (x_C, x_L) = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$ is desired to track a command signal $\mathbf{r} \in \mathbb{R}^2$, and $\mathbf{x}_I = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$ being the integral states of $\mathbf{x}_R$. Accordingly, the modified dynamics of motion for tracking become

$$
\begin{aligned}
\tilde{\mathbf{f}} &= [\tilde{x}_3, \tilde{x}_4, \mathbf{f}]^T \in \mathbb{R}^{10} \text{ and} \qquad (5.22)\\
\tilde{B} &= [0_{2\times 2}, B]^T \in \mathbb{R}^{10\times 2}. \qquad (5.23)
\end{aligned}
$$

The control objective is, for the state variable $\mathbf{x}$, to track the command $\mathbf{x}_R = (x_C, x_L) \to \mathbf{r}$, and stabilize the other states $x_2, x_4, x_5, \cdots, x_8$ [114, 244]. A significant reference command is adopted as $\mathbf{r} = (0.2, 0.7)$, being effective when $t \geq 1$, and the initial condition $\tilde{\mathbf{x}}_0 = (0, 0, 0, 0.95, 0, 0, \cdots, 0)$ is considered. Moreover, select the following weighting matrices

$$
C = I_8,\ \tilde{C} = \left[\begin{array}{c:c} I_2 & \\ \hdashline 0_{6\times 2} & \mathrm{C} \end{array}\right],\ \tilde{Q} = 100 \cdot \tilde{C}^T \tilde{C},\ R = 0.1 \cdot I_2. \qquad (5.24)
$$

Since rank(C) =dim$(\mathbf{x})$, by Theorem 2.2.1, we have that at every nonzero state $\mathcal{A}^{s\gamma}_{\mathbf{xf}} \neq \emptyset$ for $\gamma = o, d, i$ (i.e. observable, detectable, and having no unobservable mode on the imaginary axis). A step forward, a feasible SDC matrix is easily generated according to Algorithm 2.3.1, which makes the corresponding SDRE (1.19) solvable, at each nonzero state.

The simulation results are presented in Figs. 5.20-5.21, with Fig. 5.20 (resp. Fig. 5.21) including the time trajectories of the state variables $\tilde{x}_1, \cdots, \tilde{x}_6$ (resp. $\tilde{x}_7, \cdots, \tilde{x}_{10}$). Note that, as seen in (c)-(d) of Fig. 5.20, the desired tracking objective is achieved, i.e. $\mathbf{x}_R = (\tilde{x}_3, \tilde{x}_4) = (x_C, x_L) \to \mathbf{r} = (0.2, 0.7)$. However, it takes relatively longer time as compared to the results under simplified dynamics and using Nonlinear Model Predictive Control [244]. Besides, the other system states (particularly, the angular deflection of the pendulum $\tilde{x}_7 = \theta$ in Fig. 5.21) are moving within a fairly small region and stabilized eventually, except the first two dummy states $\tilde{x}_1, \tilde{x}_2$ used for the integral servomechanism of tracking. The dummy state $\tilde{x}_1$ (resp. $\tilde{x}_2$) is just the integral state of $\tilde{x}_3 = x_C$ (resp. $\tilde{x}_4 = x_L$), and as seen in (a) (resp. (b)) of Fig. 5.20, the slope is about 0.2 (resp. 0.7). This could be roughly inferred from the tracking behavior of $\tilde{x}_3 = x_C$ in (c) (resp. $\tilde{x}_4 = x_L$ in (d)), which slightly fluctuates around the target/steady state 0.2 (resp. 0.7). Note that, how to alleviate such fluctuations by exploiting the design degree of freedom (i.e. the infinitely many choices of SDC matrices) in the SDRE scheme, as parameterized by Theorems 3.2.1-3.2.2, is under investigation.

To conclude, this simulation setup successfully demonstrates the effectiveness of the tracking scheme, by implementing the SDRE controller as an integral servomechanism [12, 51, 64, 66, 67]. However, it would be more persuasive, if more theoretical support is included for such an SDRE-tracking scheme, and thus such an SDRE-tracking scheme could become more acknowledged and popular. For example, how to determine the conditions on such an SDRE-tracking scheme, such that the converging/stabilizing behavior to the target state could be guaranteed, is definitely worth investigating and serves as a future research.

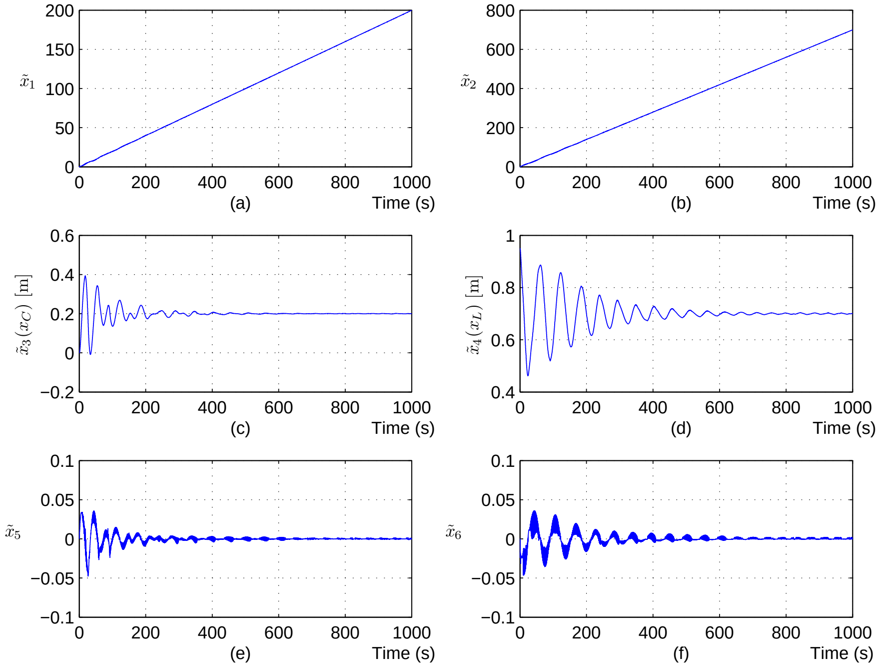

Figure 5.20: Example 5.2.3 - Time history of the system states $\tilde{x}_1, \cdots, \tilde{x}_6$.

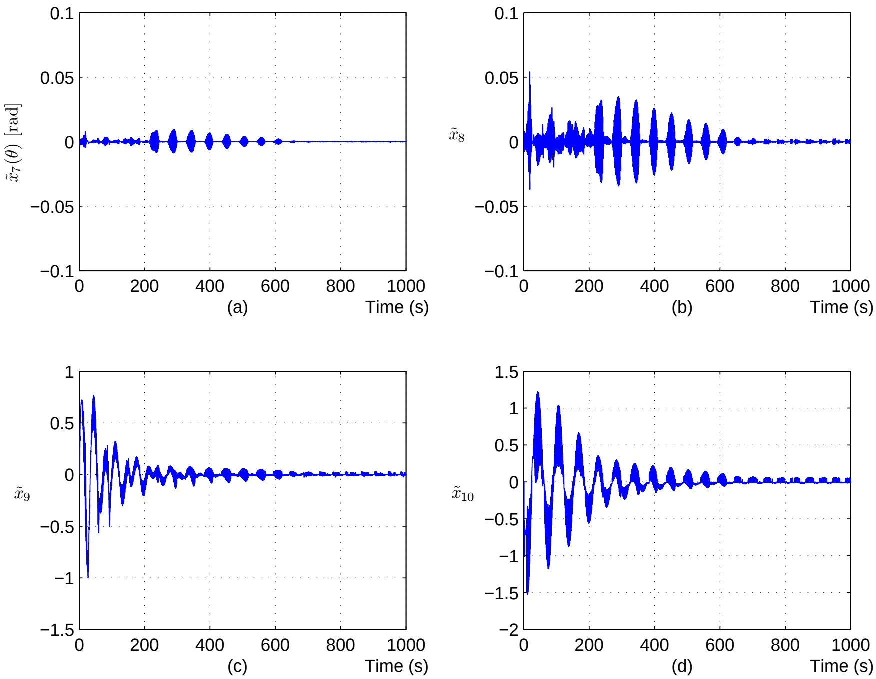

Figure 5.21: Example 5.2.3 - Time history of the system states $\tilde{x}_7, \cdots, \tilde{x}_{10}$.

5.3 Concluding Remarks

In this chapter, the effectiveness of the proposed scheme is successfully demonstrated through various examples, including significant planar cases from textbooks and real-world applications. As seen from the simulation results, the proposed scheme is required to be activated in several cases in the phase plane, such as when the system trajectory crosses some points (Example 5.2.1), some lines (Example 5.1.3), some regions (Example 5.1.4), or throughout the whole considered time horizon (Example 5.2.2). Besides, the connections to several topics of research related to the SDRE/SDDRE scheme (e.g. computational performance and solving the closed-form solution of SDDRE) are also successfully illustrated. Last but not least, in the simulation setup, a comparison of the performance with several reputed control strategies (e.g. Sontag's formula, FL, and BS) also favors the potential and importance of the proposed SDRE scheme.

Chapter 6

Closing with Final Remarks

6.1 Conclusions

In this thesis, we consider the SDRE/SDDRE scheme for nonlinear control systems, and provide more theoretical support to endorse the successful real-time applications. The proposed results are for the nonlinear time-variant systems with general orders, and are demonstrated to effectively continue the SDRE/SDDRE scheme, when the fixed SDC matrix results no solution yet the presented solvability condition is satisfied, via several benchmark examples from textbooks and significant real-world applications (namely the satellite attitude control, vector thrust control, and the tracking control of an overhead crane). Specifically, we summarize the main results as follow:

1. Theorem 2.2.1 formulates necessary and sufficient conditions for the existence of feasible SDC matrices, such that the corresponding SDRE/SDDRE (1.19/1.16) would result the solution with properties of existence, uniqueness, and positive (semi-)definiteness or not.

2. Algorithm 2.3.1 concentrates on an easy construction of feasible SDC matrices.

3. Theorem 3.1.1 (with the preliminary results) efficiently determines the solutions' property (namely existence, uniqueness, positive semi-definiteness, and positive definiteness) of the corresponding SDRE/SDDRE (1.19/1.16), given any SDC matrix. Hence Problem 1 is solved by the order reduction and the equivalent coordinate transformation (3.2).

4. Theorems 3.2.1-3.2.2 focus on the algebraic degree of freedom, by parameterizing/representing all the feasible SDC matrices, such that the corresponding SDRE/SDDRE (1.19/1.16) would result the solution with properties as existence, uniqueness, and positive (semi-)definiteness.
5. Theorem 4.5.1 considers the optimality issue of the SDRE/SDDRE scheme, with some basic and fundamental results.

Based on the above-mentioned results, it is promising to extend the studies on SDRE/SDDRE scheme to a next level, in various topics such as those described in Chapter 4 and the following section. It is expected that the proposed scheme could excite the curiosity, engage more interests and concentrate energy of the research on both schemes among the control community, for the ultimate analytical issues (e.g. global asymptotic stability and optimal control recovery) and the practical real-world applications.

6.2 Suggestions for Future Research

In several contributions (e.g. [54, 104, 113, 153]), it is pointed out that there are still interesting and worth-investigating topics related to the SDRE/SDDRE scheme. Among them, the issue of the global (asymptotical) stability seems to attract the most attention [104, 153]. Therefore, how to select appropriately the SDC matrix such that the closed-loop system is at least stabilizable would be the main concern of my future research. Additionally, the following open problems related to the SDRE/SDDRE scheme are suggested, aiming for a more established and popular SDRE/SDDRE paradigm:

1. In the SDDRE scheme (finite-time horizon nonlinear optimal control), the additional design parameter t_f makes the scheme more challenging than the SDRE scheme (infinite-time horizon nonlinear optimal control). Moreover, the specific relation between t_f with the system's property is not clear yet, although in some papers (e.g. [110, 112]), it is illustrated via both the benchmark nonlinear example (the Van der Pol's oscillator) and a more complex nonlinear problem (the spacecraft detumbling), that the more time allowed (larger t_f), the less cost required. This could be anticipated by intuition, and agrees with the counterpart in the case of the LTI system. However, regarding such a finite-time horizon, it might be difficult to (asymptotically) stabilize the general nonlinear system within a finite time, as anticipated by contributions like [10].
2. Intuitively, the performance and behavior by the SDDRE scheme might converge to those by the SDRE scheme, but at this moment no rigorous

proof is available in the literature, to the author's understanding. However, it is well known that (e.g. [110,112]), at least for linear systems, the solution of the differential Riccati equation converges to that of the algebraic Riccati equation, i.e. for LTI systems the solution of SDDRE (1.16) converges to that of SDRE (1.19), if $t_f \gg t_0$. This can be seen from the simulation demonstrations in [110,112]. In summary, if the final time t_f is set to be large, whether the SDDRE scheme converges to the SDRE scheme is worth investigating.

3. The robustness (resp. reliability) issue related to the SDRE/SDDRE scheme have been studied for years [51,54], however it needs more findings to guarantee that the SDRE/SDDRE scheme is a robust (resp. reliable) control strategy. A promising way toward these research directions is through a proper selection of the SDC matrix, based on the proposed construction/parameterization method. Note that, in Section 5.2.1, the SDRE scheme is shown to be reliable in the simulation setup. On the other hand, when combined with the reputed ISMC scheme in Section 5.2.2, the controlled system is robust to the disturbance/uncertainty appearing in the inputs.

4. In Section 4.5, there are several early-stage findings in the optimality of the SDRE/SDDRE scheme, and two main problems are suggested for the next-stage development, as below:

   - The information of the Lyapunov (cost) function is assumed to be known ahead of time, which seems to be impossible to implement. In the literature (e.g. [255]), there are several ways to generate the Lyapunov function, and it is possible to resort to these methods such that the assumption could be removed, making the SDRE/SDDRE scheme an implementable optimal control design.
   - The proper selection of the SDC matrices, as parameterized by Theorems 3.2.1-3.2.2, to satisfy the necessary optimality conditions in Theorem 4.5.1 is under investigation. Indeed, how to extract the (possibly) unique element from the design degree of freedom with infinitely-many choices ($n > 1$) is by no means and never an easy task.

5. Regarding the system's performance (such as the converging time, the control efforts, the prescribed saturation bound, the maximum overshoot, and the steady-state error), the relation between the specific SDC matrix with any property mentioned above is vague so far, and surely serves as an interesting research topic.

6. Among all successful real-time execution of SDRE/SDDRE controls, one significant demonstration is the fully embedded *twelve-state* SDRE

controller of an UAV [29], being able to compute the SDRE control law in 14 ms (70 Hz) using the 300MHz Geode GX1 microprocessor and MIT's software package. Promisingly, it is possible to adopt the findings in Sec. 4.2 to solve the SDRE alternatively, which accordingly require minor modifications of the software package, and may improve the numerical performance.

7. From the numerical point of view, how to pick out the optimal SDC matrix/matrices (as parameterized in Section 3.2) corresponding to the best numerical quality (such as the least computation effort, the most accurate solution, and the most reliable/least sensitive result) definitely needs more investigations, which opens a new avenue in the SDRE/SDDRE research.

8. Inherently, the SDRE/SDDRE control technique shares several symmetries with the design of Linear Quadratic Regulator. Still, in the process industry it also shares similarities with the Model Predictive Control [145]. Furthermore, under certain cases in Example 5.1.1, the control laws designed by Back-Stepping, Sontag's formula, and the SDRE scheme are identical. As such, by taking advantage of the design degree of freedom (namely both the tuning of weighting matrices and the infinitely many feasible SDC matrices), some intuitive and interesting questions arise:

   - does the SDRE/SDDRE scheme share more similarities with the above-mentioned control designs?
   - does the SDRE/SDDRE scheme share any similarity with other control designs?
   - is it possible to include any other control design in the diversity and flexibility of the SDRE/SDDRE scheme?

9. As demonstrated in Example 5.2.2 and [140], the SDRE scheme could be robust to disturbances/uncertainties, by combining with the Integral-type Sliding Mode Control. For other beneficial properties (such as reliability), is it possible to combine with any other control design to gain such utility?

10. The tracking scheme, by implementing the SDRE controller as an integral servomechanism [12, 51, 64, 66, 67], is described in Section 4.3 and successfully demonstrated in Example 5.2.3. However, if more theoretical support could be included for such an SDRE-tracking scheme, then such an SDRE-tracking scheme could become more acknowledged and popular. For example, how to determine the conditions on such an SDRE-tracking scheme, such that the converging/stabilizing behavior to the target state could be guaranteed, is definitely worth investigating and serves as a

future research. Toward this direction, [189] certainly provides several essential and preliminary results.

Appendix A

In this chapter/appendix, we denote c and s as the cos and sin functions, respectively.

A.1 Justification of $A_p = arg \min_{A \in \mathcal{A}_{\mathbf{xf}}} ||A||_F$

By the property of "submultiplicative" for the Frobenius norm, i.e. $||A\mathbf{x}||_F \leq ||A||_F \cdot ||x||_F$, together with $\mathbf{f} = A\mathbf{x}$ for all $A \in \mathcal{A}_{\mathbf{xf}}$ as in the SDRE/SDDRE scheme, then we have

$$
\begin{aligned}
||A||_F &\geq \frac{||A\mathbf{x}||_F}{||x||_F} \\
&= \frac{||\mathbf{f}||_F}{||\mathbf{x}||_F} \\
&= \frac{\left(\sum_{i=1}^{n} f_i^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}}.
\end{aligned}
$$

On the other hand, the considered matrix A_p as defined in the beginning of Chapter 3 is detailed as below

$$\begin{aligned} A_p &= \frac{\mathbf{f}\mathbf{x}^T}{||x||^2} \\ &= \begin{bmatrix} \frac{f_1 x_1}{||\mathbf{x}||^2} & \cdots & \frac{f_1 x_n}{||\mathbf{x}||^2} \\ & \vdots & \\ \frac{f_n x_1}{||\mathbf{x}||^2} & \cdots & \frac{f_n x_n}{||\mathbf{x}||^2} \end{bmatrix}, \end{aligned}$$

with the Frobenius norm of A_p given as

$$\begin{aligned} ||A_p||_F &= \left(\frac{f_1^2 x_1^2}{||x||^4} + \cdots + \frac{f_1^2 x_n^2}{||x||^4} + \cdots + \frac{f_n^2 x_1^2}{||x||^4} + \cdots + \frac{f_n^2 x_n^2}{||x||^4} \right)^{\frac{1}{2}} \\ &= \left(\frac{f_1^2}{||x||^4} \cdot ||\mathbf{x}||^2 + \cdots + \frac{f_n^2}{||x||^4} \cdot ||\mathbf{x}||^2 \right)^{\frac{1}{2}} \\ &= \frac{\left(\sum_{i=1}^{n} f_i^2 \right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}} \end{aligned}$$

, therefore the result follows.

A.2 SDC matrix in Example 5.2.1

Denote I_1, I_2, and I_3 as $\frac{I_y - I_z}{I_x}$, $\frac{I_z - I_x}{I_y}$, and $\frac{I_x - I_y}{I_z}$, respectively. Then we can factorize the drift term into the adopted SDC matrix, and the elements are described as $A(\mathbf{x}, t) = [a_{ij}(\mathbf{x}, t)]$, where

$$
\begin{aligned}
a_{1j} &= 0, j = 1, 2, 3, 5, 6; \text{ and a}_{14} = 1. \\
a_{2j} &= 0, j = 1, 2, 3, 4, 6; \text{ and a}_{15} = 1. \\
a_{3j} &= 0, j = 1, 2, 3, 4, 5; \text{ and a}_{16} = 1. \\
a_{41} &= \frac{1}{4} I_1 \omega_0 x_5 \frac{cx_1 - 1}{x_1} sx_3 sx_2 + \frac{1}{3} I_1 \omega_0 x_5 cx_3 \frac{sx_1}{x_1} + I_1 \omega_0 x_6 \frac{cx_1 - 1}{x_1} \\
&\quad + \frac{1}{6} I_1 \omega_0^2 s(2x_3) \frac{c^2 x_1 - 1}{x_1} sx_2 + \frac{1}{4} I_1 \omega_0^2 \frac{sx_1}{x_1} + \frac{1}{4} I_1 \omega_0^2 c^2 x_3 \frac{s(2x_1)}{x_1} \\
&\quad - \frac{1}{4} I_1 \omega_0 x_6 sx_3 sx_2 \frac{sx_1}{x_1} + \frac{1}{6} I_1 \omega_0^2 s^2 x_2 s^2 x_3 \frac{s(2x_1)}{x_1} \\
&\quad - \frac{1}{6} I_1 \omega_0^2 s(2x_3) sx_2 \frac{s^2 x_1}{x_1} - \frac{3}{4} I_1 \omega_0^2 c^2 x_2 \frac{s(2x_1)}{x_1} - \frac{3}{4} I_1 \omega_0^2 \frac{s(2x_1)}{x_1}. \\
a_{42} &= \omega_0 x_6 \frac{cx_2 - 1}{x_2} - \frac{1}{3} \omega_0 x_5 sx_3 \frac{sx_2}{x_2} + \frac{1}{4} I_1 \omega_0 x_5 cx_1 sx_3 \frac{sx_2}{x_2} \\
&\quad + \frac{1}{6} I_1 \omega_0^2 s(2x_3) \frac{sx_2}{x_2} + \frac{1}{6} I_1 \omega_0^2 sx_3 \frac{sx_2}{x_2} - \frac{1}{4} I_1 \omega_0 x_6 sx_3 \frac{sx_2}{x_2} sx_1 \\
&\quad - \frac{1}{6} I_1 \omega_0^2 \frac{s^2 x_2}{x_2} s^2 x_3 s(2x_1) - \frac{1}{6} I_1 \omega_0^2 s(2x_3) \frac{sx_2}{x_2} s^2 x_1 \\
&\quad - \frac{3}{4} I_1 \omega_0^2 \frac{c^2 x_2 - 1}{x_2} s(2x_1).
\end{aligned}
$$

$$
\begin{aligned}
a_{43} &= \omega_0 x_6 \frac{cx_3 - 1}{x_3} - \frac{1}{3}\omega_0 x_5 \frac{sx_3}{x_3} sx_2 + \frac{1}{4} I_1 \omega_0 x_5 cx_1 \frac{sx_3}{x_3} sx_2 \\
&\quad + I_1 \omega_0 \frac{1}{3} x_5 \frac{cx_3 - 1}{x_3} sx_1 + I_1 \omega_0 x_6 \frac{cx_3 - 1}{x_3} + \frac{1}{6} I_1 \omega_0^2 \frac{s(2x_3)}{x_3} c^2 x_1 sx_2 \\
&\quad + \frac{1}{4} I_1 \omega_0^2 \frac{c^2 x_3 - 1}{x_3} s(2x_1) - \frac{1}{4} I_1 \omega_0 x_6 \frac{sx_3}{x_3} sx_2 sx_1 \\
&\quad - \frac{1}{6} I_1 \omega_0^2 s^2 x_2 \frac{s^2 x_3}{x_3} s(2x_1) - \frac{1}{6} I_1 \omega_0^2 \frac{s(2x_3)}{x_3} sx_2 s^2 x_1. \\
a_{44} &= 0. \\
a_{45} &= -\frac{1}{3}\omega_0 sx_3 sx_2 + \frac{1}{2} I_1 x_6 + \frac{1}{4} I_1 \omega_0 (cx_1 sx_3 sx_2 + sx_3 sx_2) \\
&\quad + \frac{1}{3} I_1 \omega_0 (cx_3 sx_1 + sx_1). \\
a_{46} &= \omega_0 [1 + (cx_3 - 1)(cx_2 - 1)] + \frac{1}{2} I_1 x_5 \\
&\quad + I_1 \omega_0 [(cx_3 - 1)(cx_1 - 1) + 1] - \frac{1}{4} I_1 \omega_0 sx_3 sx_2 sx_1. \\
a_{51} &= \frac{1}{3}\omega_0 x_6 sx_3 \frac{cx_1 - 1}{x_1} + \frac{1}{3}\omega_0 x_4 cx_3 \frac{sx_1}{x_1} + \frac{1}{4}\omega_0 x_6 cx_3 sx_2 \frac{sx_1}{x_1} \\
&\quad + \frac{1}{4}\omega_0 x_5 sx_3 cx_2 \frac{sx_1}{x_1} + \frac{1}{4}\omega_0 x_4 sx_3 sx_2 \frac{cx_1 - 1}{x_1} \\
&\quad + \frac{1}{4} I_2 \omega_0 x_4 \frac{cx_1 - 1}{x_1} sx_3 sx_2 + \frac{1}{3} I_2 \omega_0 x_4 cx_3 \frac{sx_1}{x_1} \\
&\quad - \frac{1}{6} I_2 \omega_0^2 sx_2 s^2 x_3 \frac{cx_1 - 1}{x_1} - \frac{1}{6} I_2 \omega_0^2 cx_2 \frac{sx_1}{x_1} s(2x_3) \\
&\quad - \frac{1}{6} I_2 \omega_0^2 \frac{sx_1}{x_1} s(2x_3) + \frac{3}{4} I_2 \omega_0^2 s(2x_2) \frac{cx_1 - 1}{x_1}.
\end{aligned}
$$

$$\begin{aligned}
a_{52} &= \frac{1}{4}\omega_0 x_6 cx_3 \frac{sx_2}{x_2} sx_1 + \frac{1}{4}\omega_0 x_5 sx_3 \frac{cx_2 - 1}{x_2} sx_1 + \frac{1}{4}\omega_0 x_4 sx_3 \frac{sx_2}{x_2} cx_1 \\
&+ \frac{1}{4} I_2 \omega_0 x_4 cx_1 sx_3 \frac{sx_2}{x_2} - \frac{1}{3} I_2 \omega_0 x_6 sx_3 \frac{cx_2 - 1}{x_2} \\
&- \frac{1}{6} I_2 \omega_0^2 \frac{s(2x_2)}{x_2} (s_3^x cx_1 + s_3^x) - \frac{1}{6} I_2 \omega_0^2 \frac{cx_2 - 1}{x_2} sx_1 s(2x_3) \\
&+ \frac{3}{4} I_2 \omega_0^2 \frac{s(2x_2)}{x_2} (cx_1 + 1). \\
a_{53} &= \frac{1}{3}\omega_0 x_6 \frac{sx_3}{x_3} cx_1 + \frac{1}{3}\omega_0 x_4 \frac{cx_3 - 1}{x_3} sx_1 + \frac{1}{4}\omega_0 x_5 \frac{cx_3 - 1}{x_3} sx_2 sx_1 \\
&+ \frac{1}{4}\omega_0 x_5 \frac{sx_3}{x_3} cx_2 sx_1 + \frac{1}{4}\omega_0 x_4 \frac{sx_3}{x_3} sx_2 cx_1 + \frac{1}{4} I_2 \omega_0 x_4 cx_1 \frac{sx_3}{x_3} sx_2 \\
&+ \frac{1}{3} I_2 \omega_0 x_4 \frac{cx_3 - 1}{x_3} sx_1 - \frac{1}{3} I_2 \omega_0 x_6 \frac{sx_3}{x_3} cx_2 \\
&- \frac{1}{6} I_2 \omega_0^2 s(2x_2) \frac{s^2 x_3}{x_3} cx_1 - \frac{1}{6} I_2 \omega_0^2 cx_2 sx_1 \frac{s(2x_3)}{x_3}. \\
a_{54} &= \frac{1}{3}\omega_0 (cx_3 sx_1 + sx_1) + \frac{1}{4}\omega_0 (sx_3 sx_2 cx_1 + sx_3 sx_2) \\
&+ \frac{1}{4} I_2 \omega_0 (cx_1 sx_3 sx_2 + sx_3 sx_2) + \frac{1}{3} I_2 \omega_0 (cx_3 sx_1 + sx_1). \\
a_{55} &= \frac{1}{4}\omega_0 (sx_3 cx_2 sx_1 + sx_3 sx_1). \\
a_{56} &= \frac{1}{3}\omega_0 (sx_3 cx_1 + sx_3) + \frac{1}{4}\omega_0 (cx_3 sx_2 sx_1 + sx_2 sx_1) \\
&+ \frac{1}{2} I_2 x_6 x_4 + \frac{1}{2} I_2 x_4 - \frac{1}{4} I_2 \omega_0 (sx_3 cx_2 + sx_3).
\end{aligned}$$

$$
\begin{aligned}
a_{61} &= \frac{1}{4}\omega_0 x_4 \frac{sx_1}{x_1} sx_2 sx_3 - \omega_0 x_6 \frac{cx_1 - 1}{x_1}(cx_3 - 1) sx_2 - \omega_0 x_5 sx_3 \frac{cx_1}{x_1} \\
&+ \frac{1}{3}\omega_0 x_6 sx_3 \frac{sx_1}{x_1} - \omega_0 x_4 \frac{cx_1 - 1}{x_1} + I_3 \omega_0 x_4 \frac{cx_1 - 1}{x_1} \\
&- \frac{1}{4} I_3 \omega_0 x_4 sx_3 sx_2 \frac{sx_1}{x_1} - \frac{1}{2} I_3 \omega_0^2 s(2x_3) \frac{cx_1 - 1}{x_1} \\
&+ \frac{1}{6} I_3 \omega_0^2 s^2 x_3 \frac{sx_1}{x_1} s(2x_2) - \frac{3}{4} I_3 \omega_0^2 s(2x_2) \frac{sx_1}{x_1}. \\
a_{62} &= \frac{1}{4}\omega_0 x_4 sx_1 \frac{sx_2}{x_2} sx_3 - \omega_0 x_6 (cx_1 - 1) \frac{sx_2}{x_2} \\
&- \omega_0 x_5 sx_3 (cx_1 - 1) \frac{cx_2 - 1}{x_2} - \frac{1}{4} I_3 \omega_0 x_4 sx_3 \frac{sx_2}{x_2} sx_1 \\
&- \frac{1}{3} I_3 \omega_0 x_5 sx_3 \frac{cx_2 - 1}{x_2} - \frac{1}{2} I_3 \omega_0^2 s(2x_3) \frac{cx_2 - 1}{x_2} \\
&+ \frac{1}{6} I_3 \omega_0^2 s^2 x_3 sx_1 \frac{sx_2}{x_2} - \frac{3}{4} I_3 \omega_0^2 \frac{s(2x_2)}{x_2} sx_1. \\
a_{63} &= \frac{1}{4}\omega_0 sx_1 sx_2 \frac{sx_3}{x_3} - \omega_0 x_6 \frac{cx_3 - 1}{x_3} sx_2 - \omega_0 x_5 \frac{sx_3}{x_3}(cx_2 - 1) \\
&+ \frac{1}{3}\omega_0 x_6 \frac{sx_3}{x_3} sx_1 - \omega_0 x_4 \frac{cx_3 - 1}{x_3} + I_3 \omega_0 x_4 \frac{cx_3 - 1}{x_3} \\
&- \frac{1}{4} I_3 \omega_0 x_3 \frac{sx_3}{x_3} sx_2 sx_1 - \frac{1}{2} I_3 \omega_0^2 \frac{s(2x_3)}{x_3} [(cx_2 - 1)(cx_1 - 1) + 1] \\
&+ \frac{1}{6} I_3 \omega_0^2 \frac{s^2 x_3}{x_3} sx_1 s(2x_2). \\
a_{64} &= \frac{1}{4}\omega_0 sx_1 sx_2 sx_3 - \omega_0 [(cx_3 - 1)(cx_1 - 1) + 1] + \frac{1}{2} I_3 x_5 \\
&+ I_3 \omega_0 [(cx_3 - 1)(cx_1 - 1) + 1] - \frac{1}{4} I_3 \omega_0 sx_3 sx_2 sx_1. \\
a_{65} &= -\omega_0 sx_3 + \frac{1}{2} I_3 x_4 - \frac{1}{3} I_3 \omega_0 \frac{sx_3}{x_3} cx_2 x_3 \frac{1}{3} I_3 \omega_0 (sx_3 cx_2 + sx_3). \\
a_{66} &= -\omega_0 sx_2 + \frac{1}{3}\omega_0 sx_3 sx_1.
\end{aligned}
$$

A.3 SDC matrix in Example 5.2.2

Note that, in this section, the trivially zero entries are not specified. And the following entries are adopted for both SDC matrices in Section A.3.1-A.3.2.

$$a_{12} = a_{34} = a_{56} = 1,\ a_{77} = -\lambda.$$

A.3.1 SDC matrix containing the most trivial zeros

$$\begin{aligned}
a_{22} &= \frac{1}{\alpha - \delta s^2\phi_2}\left[2\left(\frac{\delta x_4}{r_f}\right)s\phi_2 c\phi_2 - \left(\frac{J_m\Omega}{r_f}\right)sx_5\right]. \\
a_{43} &= \left[-\left(\frac{\delta x_2^2}{r_f\beta}\right)c\phi_2 - \kappa\right]\frac{s\phi_2}{x_3} - \gamma\frac{c\phi_2 - 1}{x_3}. \\
a_{47} &= \frac{-\gamma}{x_7}. \\
a_{65} &= \left(\frac{J_m\Omega x_2}{Jr_f} - \frac{m_f gl}{J}\right)\frac{sx_5}{x_5}.
\end{aligned}$$

A.3.2 SDC matrix containing the fewest trivial zeros

$$a_{22} = \frac{1}{\alpha - \delta s^2\phi_2}\left[\frac{2}{3}\left(\frac{\delta x_4}{r_f}\right)s\phi_2 c\phi_2 - \frac{1}{2}\left(\frac{J_m\Omega}{r_f}\right)sx_5\right].$$

$$a_{23} = \frac{1}{\alpha - \delta s^2\phi_2}\left[\frac{2}{3}\left(\frac{\delta x_2 x_4}{r_f}\right)c\phi_2\frac{s\phi_2}{x_3}\right].$$

$$a_{24} = \frac{1}{\alpha - \delta s^2\phi_2}\left[\frac{2}{3}\left(\frac{\delta x_2}{r_f}\right)s\phi_2 c\phi_2\right].$$

$$a_{25} = \frac{1}{\alpha - \delta s^2\phi_2}\left[-\frac{1}{2}\left(\frac{J_m\Omega x_2}{r_f}\right)\frac{sx_5}{x_5}\right].$$

$$a_{42} = -\frac{1}{2}\left(\frac{\delta x_2}{r_f\beta}\right)c\phi_2 s\phi_2.$$

$$a_{43} = \left[-\frac{1}{2}\left(\frac{\delta x_2^2}{r_f\beta}\right)c\phi_2 - \kappa\right]\frac{s\phi_2}{x_3} - \gamma\frac{c\phi_2 - 1}{x_3}.$$

$$a_{47} = \frac{-\gamma}{x_7}.$$

$$a_{62} = \frac{1}{2}\left(\frac{J_m\Omega}{Jr_f}\right)sx_5.$$

$$a_{65} = \left(\frac{J_m\Omega x_2}{2Jr_f} - \frac{m_f gl}{J}\right)\frac{sx_5}{x_5}.$$

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Curriculum Vitae

Li-Gang (Charles) Lin was born on November 24, 1984 in Changhua, Taiwan. He received the B.S. degree and M.S. degree in Electrical Control Engineering (ECE) from National Chiao Tung University (NCTU), Hsinchu, Taiwan, in 2008 and in 2010, respectively. He is currently working towards his joint/double Ph.D. degrees from ESAT (Department of Electrical Engineering) of Katholieke Universiteit Leuven, Belgium, and Institute of Electrical Control Engineering of NCTU. His research is supported in part by Belgian Federal Science Policy Office: IUAP P7 (Dynamical systems, control and optimization, DYSCO); Research Council KUL: PFV/10/002 Optimization in Engineering Center (OPTEC); and National Science Council, Taiwan, ROC, under Grant NSC 100-2221-E-009-026-MY2. His research interests include nonlinear control systems, state-dependent (differential) Riccati equation, variable structure systems, industrial electronics, reliable and robust control.

List of publications

(ordered by year)

1. Y.-W. Liang, L.-W. Ting, L.-G. Lin, and Y.-T. Wei, "Study of Reliable Control via an Integral-Type SMC Scheme," in *Proceedings of 2009 CACS International Automatic Control Conference, Taipei, Taiwan*, Nov. 2009.
2. Y.-W. Liang, Y.-T. Wei, D.-C. Liaw, C.-C. Cheng, and L.-G. Lin, "A study of SDRE and ISMC combined scheme with application to vehicle brake control," in *SICE Annual Conference, Proceedings of.* IEEE, 2010, pp. 497–502.
3. L.-G. Lin, "Reliable nonlinear control via combining SDRE and ISMC approaches," Master's thesis, National Chiao Tung University, Taiwan, Republic of China, 2010.
4. Y.-W. Liang and L.-G. Lin, "On factorization of the nonlinear drift term for SDRE approach," in *Proceedings of the 18th IFAC World Congress, Milano, Italy*, vol. 18, no. 1, Aug. 2011, pp. 9607–9612.
5. Y.-W. Liang, L.-W. Ting, and L.-G. Lin, "Study of Reliable Control Via an Integral-Type Sliding Mode Control Scheme," *Industrial Electronics, IEEE Transactions on*, vol. 59, no. 8, pp. 3062–3068, Aug. 2012.
6. Y.-W. Liang, J.-Y. Chen, and L.-G. Lin, "A guidance law design using the combination of ISMC and SDRE schemes," in *IEEE International Conference on System Science and Engineering*, Aug. 2013, pp. 63–67.
7. Y.-W. Liang and L.-G. Lin, "Analysis of SDC matrices for successfully implementing the SDRE scheme," *Automatica*, vol. 49, no. 10, pp. 3120–3124, 2013.
8. L.-G. Lin, J. Vandewalle, and Y.-W. Liang, "Flexibility of the state-dependent (differential) Riccati equation scheme for nonlinear systems with arbitrary orders," under review.

9. ——, “Vector thrust control via the state-dependent Riccati equation scheme,” under review.

The following list some information and reviews about the above publications:

1. received the Excellent Paper Award.

4. (EI conference)
   It was invited by the chairman of the SDRE session on the IFAC congress, Prof. Tayfun Çimen.

5. (SCI, EI, 13IF: 6.5, rank: 2/247(0.8%), 1/57(1.75%), and 2/59(3.38%))

7. (SCI, EI, 08IF: 3.18, rank: 3/53(5.66%) and 19/229(8.29%), a journal of IFAC)
   It is the only contribution of the SDRE scheme published in *Automatica* since 2001, and thus may be regarded as a breakthrough. Note that for *IEEE Transactions on Automatic Control*, [222] (2003) is the last SDRE contribution. It was supported by the National Science Council, Taiwan, under Grants 99-2218-E-009-004, 100-2221-E-009-026-MY2, and 101-2623-E-009-005-D.

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